

## CORRECTIONS TO

### A RADICAL APPROACH TO LEBESGUE'S THEORY OF INTEGRATION

**Page 10**, line 4, "Propogation" should be "Propagation"

**Page 12**, Footnote 5, "Weierstrass's" should be "Weierstrass."

**Page 23**, line before displayed quote, it was to Riemann's father, rather than to Richard Dedekind, to which this letter was addressed

**Page 23**, footnote, the Hochkirchen reference should be to page 263 rather than 261.

**Page 24**, line 8, "oscillirendend" should be "oszillierenden."

**Page 27**, Theorem 2.1, line 4, change "We can find a  $\delta$  response" to "We can find a response  $\delta$ "

**Page 30**, last set of displayed equations, second last line should be

$$= \lim_{\epsilon_1 \rightarrow 0^-} \left( -2|\epsilon_1|^{1/2} + 2 \right) + \lim_{\epsilon_2 \rightarrow 0^+} \left( 2 - 2|\epsilon_2|^{1/2} \right)$$

**Page 38**, first 6 lines (the justification that  $w(x)$  is not differentiable at  $a_N$ ). This justification is not correct. The correct justification is given by:

We first observe that  $\sum_{n=1}^{N-1} k^n h(x - a_n)$  is differentiable at  $x = a_N$ . Therefore,  $w$  is differentiable at  $x = a_N$  if and only if  $\sum_{n=N}^{\infty} k^n h(x - a_n)$  is differentiable at  $x = a_N$ . If this sum is differentiable, then the oscillation of the average rate of change between  $a_N$  and  $x$ , as a function of  $x$ , can be made arbitrarily close to 0 by taking  $x$  sufficiently close to (but not equal to)  $a_N$ . But this average rate of change can be written as a sum of two fractions:

$$\frac{k^N h(x - a_N) - k^N h(0)}{x - a_N} + \frac{\sum_{n=N+1}^{\infty} k^n h(x - a_n) - \sum_{n=N+1}^{\infty} k^n h(a_N - a_n)}{x - a_N}.$$

The first fraction,

$$\frac{k^N h(x - a_N) - k^N h(0)}{x - a_N} = k^N \left( 1 + \frac{1}{2} \sin \left( \frac{1}{2} \ln [(x - a_N)^2] \right) \right),$$

oscillates between  $k^N/2$  and  $3k^N/2$  over any interval that contains  $a_N$ . By the mean value theorem,

$$\left| \frac{\sum_{n=N+1}^{\infty} k^n h(x - a_n) - \sum_{n=N+1}^{\infty} k^n h(a_N - a_n)}{x - a_N} \right| < 2 \sum_{n=N+1}^{\infty} k^n = \frac{2k^{N+1}}{1 - k},$$

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*Date:* September 14, 2011.

and therefore the oscillation of the second fraction is strictly less than  $4k^{N+1}/(1-k)$ . If  $k < 1/5$ , then the oscillation of the second fraction is bounded above by an amount strictly less than the oscillation of the first fraction, and therefore the sum of their oscillations cannot be made arbitrarily small.

**Page 39**, line 4, “Exercises 2.2.4–2.2.5” should be “Exercises 2.2.4–2.2.9”

**Page 42**, line 2 under **Hankel’s Innovations**, “Funckionen” should be “Funktionen.”

**Page 46**, paragraph 3, line 6: “funzioni” should be “funzioni”  
 paragraph 4, line 2, “Riehen” should be “Reihen”

**Page 47**, third definition box, “A  $S$  set is type  $n$ ” should be “A set  $S$  is type  $n$ ”

**Page 49**, Exercise 2.3.9: “the oscillation of  $r$ ” should be “the oscillation of  $g$ ”

**Page 50**, Exercise 2.3.13,  $\sin(1/f_1(x))$  and  $\sin(1/f_2(x))$  should be  $\sin(1/f_1(x))$  and  $\sin(1/f_2(x))$

**Page 50**, Exercise 2.3.15, “Show that if  $C \in \mathcal{C}$  is an finite cover” should be “Show that if  $C \in \mathcal{C}$  is a finite cover”

**Page 55**, last paragraph of proof of Theorem 3.1, change “then pick any  $f(x) \in T$  and any  $\epsilon$ ” to “then pick any  $x$  and any  $\epsilon$ ”

**Page 68**, third full paragraph, line 3, “Functionenlehre” should be “Funktionenlehre”

**Page 70**, Exercise 3.2.12, the displayed summations

$$\sum_{n=1}^{\infty} \frac{A_n}{b_n} \quad \text{and} \quad \sum_{n=1}^{\infty} A_n$$

should be

$$\sum_{n=1}^{\infty} \frac{A_n}{b_n} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

**Page 79** Exercise 3.3.4, in the first line, “each of following” should be “each of the following”. In 2. “ $\mathbb{R}$  and set” should be “ $\mathbb{R}$  and the set”. In 5. “ $\mathbb{R}$  and the set all pairs” should be “ $\mathbb{R}$  and the set of all pairs”

**Page 81** fifth line from bottom: “there are also nowhere dense set that” should be “there are also nowhere dense sets that”.

**Page 86**, last paragraph, change

This function has the curious property that it is a nonconstant function with a derivative that exists and is zero at every point except at a set with outer content zero.

to

This function has the curious property that it is a continuous nonconstant function with a derivative that exists and is zero at every point except at a set with outer content zero.

**Page 89**, second paragraph of section 4.2. Luca La Rocca has sent the following information about those Italian professors who refused to sign the oath of allegiance to the Fascist government:

You may like to know that there were at least 12 (not eleven) Italian professors who “forgot” to sign “yet another piece of paper” on that occasion; I like to tell the story in this way, because my feeling is that it is in this way that these things creep up... and in this way we should expect them to show up in the future... hopefully in a worst case scenario. Giorgio Boatti in his “Preferirei di no” (I’d rather prefer not) inspired by Herman Melville’s

[http://en.wikipedia.org/wiki/Bartleby,\\_the\\_Scrivener](http://en.wikipedia.org/wiki/Bartleby,_the_Scrivener)

lists Ernesto Buonaiuti, Mario Carrara, Gaetano De Sanctis, Giorgio Er-rera, Giorgio Levi Della Vida, Fabio Luzzatto, Piero Martinetti, Bartolo Nigrisoli, Francesco Ruffini, Edoardo Ruffini, Lionello Venturi and Vito Volterra. According to Boatti it is Fabio Luzzatto who is usually forgotten. A few more names appear on Wikipedia’s

[http://it.wikipedia.org/wiki/Giuramento\\_di\\_fedelt\\_al\\_Fascismo](http://it.wikipedia.org/wiki/Giuramento_di_fedelt_al_Fascismo)

namely Giuseppe Antonio Borghese, Enrico Presutti, Aldo Capitini and Antonio De Viti De Marco. The first two names are supported by a letter from their heirs following a review of Boatti’s book in an Italian newspaper

<http://ricerca.repubblica.it/repubblica/archivio/repubblica/2000/04/22/professori-che-rifiutarono-il-giuramento.html>

the third one was apparently sacked (about the same time) for a different kind of conflict with the regime, while the last one resigned to avoid swearing the oath.

**Page 91**, graph of  $SVC(4)$ : The top row of line segments should each have their middle quarters removed.

**Page 98**, last paragraph, line 6, “Functionenlehre” should be “Funktionenlehre”

**Page 112**, second line from bottom: “Dirichet’s function” should be “Dirichlet’s function”.

**Page 114**, line 1, “The function  $g$  of Exercise 1.1.5 (p. 15)” should be “The function  $m$  of Exercise 2.1.14 (p. 32)”

**Page 126**, third line from top: “well-fined” should be “well-defined”.

**Page 149**, Exercise 5.3.10. Second line. The inequality “ $m_\epsilon(S) < m(U) + \epsilon$ ” should be “ $m(U) < m_\epsilon(S) + \epsilon$ .”

**Page 149**, Exercise 5.3.11. part 4. The closed set  $C$  should be contained in  $S$ : there is a closed set  $C \subseteq S$  such that  $m_\epsilon(S - C) < \epsilon$ .

**Page 150**, Exercise 5.3.15, “the supremum and the infimum” should be “the lim sup and the lim inf”

**Page 157**, Exercises 5.4.2 and 5.4.3: For both exercises, we need to assume that the set in question is measurable. Both first sentences should begin, “Show that any measurable set of positive measure contains . . .”

**Page 196**, Proof of Theorem 6.24, the symbol indicating the end of the proof should come just before the next subhead: **Limits of Step Functions**

**Page 203**, End of second paragraph: “functional theorem of calculus” should be “fundamental theorem of calculus.”

**Page 205**, line 2, second set of equalities should be

$$D^- f(0) = D_- f(0) = -1.$$

Paragraph 3, line 3, “particular” should be “particular”

**Page 208**, line 4 from bottom, “ $f = f - (T - f)$ ” should be “ $f = T - (T - f)$ .”

**Page 220**, Equation (7.16) should be

$$g(t_x) - g(s_x) > N(t_x - s_x).$$

**Page 232**, First line below equation (7.19), the sentence should begin, “On the other hand, since  $F'$  exists at  $x \in E_p^q$ , then”

**Page 265**, The case  $p = \infty$  is not quite done. We need to show that  $f_n$  converges to  $f$  in the  $L^\infty$  norm. Given  $\epsilon > 0$ , choose  $N$  so that  $m, n \geq N$  implies that  $\|f_m - f_n\|_\infty < \epsilon/2$ . For  $x \in [a, b] - F$ ,  $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \epsilon/2$ . Since  $f_n(x)$  converges pointwise to  $f(x)$ , we also know that there is an  $M_x \geq N$  such that  $n \geq M_x$  implies that  $|f_n(x) - f(x)| < \epsilon/2$ . Therefore, for every  $x \in [a, b] - F$ , and  $n \geq N$ ,

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{M_x}(x)| + |f_{M_x}(x) - f(x)| < \epsilon.$$

It follows that  $f_n$  converges uniformly to  $f$  on  $[a, b] - F$ , and therefore it converges in the  $L^\infty$  norm to  $f$  on  $[a, b]$ .

**Page 295**, line 6 from bottom,  $|x_j - x_{j-1}| < \delta(x_j^*) < \inf_{x \in G_k^C} |x_k^* - x|$  should be  $|x_j - x_{j-1}| < \delta(x_j^*) < \inf_{x \in G_{k_j}^C} |x_j^* - x|$ . Also note that the inequalities at the top of

page 296 remain valid even when several tags  $x_j^*$  share the same  $E_k$ . It works because each of them generates a separate contribution within the same set  $(G_k - E_k)$ .

**Page 297**, line 18, “most useful” should be “more useful”

**Page 300**, Hint for exercise 1.2.6. First sentence should read, “Let  $S_n = \sup_{k \geq n} a_k$ .”

**Page 303**, Hint for exercise 3.3.10, line 4. Both occurrences of  $A'$  should just be  $A$ . This line should read: “ $n$  so that  $\psi^n(a) \in B - A$ , then  $a$  is mapped to  $\psi(a)$ . If  $a$  is not in  $A$  or there ...”

**Page 309:**, Hint for exercise 6.4.9. The two numbers  $a, b \in \mathcal{N}$  need to be on the same side of  $x$ .

**Page 309:**, Hint for exercise 6.4.14. Change the hint to read:

For each  $n \in \mathbb{N}$ , choose a measurable set  $E_n$  with measure less than  $1/n$  for which  $f$  is continuous relative to  $[a, b] - E_n$ . Show that there is an open set  $U_n$  such that

$$U_n - E_n \subseteq \{x \mid f(x) > c\} \subseteq U_n \cup E_n.$$

Use exercise 5.3.9.

**Page 319:**, Hochkirchen, T. 2003. Page numbers should be 261–290.

**Thanks to:** Hossein Alidaee, Jorge Banuelos, Elise Delmas, Patrice Goyer, Sean Hickey, Peter Horton, Luca La Rocca, Jim Leonard, Anu Mishra, Stylianos Pezaris, Fred Rickey, Bill Rozzi, Allen Rubenstein, Lou Talman, and Hao Zou.