Corrected Proof of Lemma 6.3*

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Lemma 6.3 (Continuity on $[a, b] \implies$ Unif. Continuity). *If f is continuous over the closed and bounded interval $[a, b]$, then it is uniformly continuous over this interval.*

Proof: We assume that f is not uniformly continuous on $[a, b]$ but that it is continuous at every point of $[a, b]$ and show that this leads to a contradiction.

To say that f is not uniformly continuous over $[a, b]$ means that there is some $\epsilon > 0$ for which there is no uniform response, no single δ that works at every point x in $[a, b]$. This, in turn, means that given any $\delta > 0$, we can always find an $x \in [a, b]$ and another point $y \in [a, b]$ such that $0 < |x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

We choose an $\epsilon > 0$ for which there is no uniform response to 2ϵ and, for each $n \in \mathbb{N}$, choose x_n, y_n in $[a, b]$ such that $|y_n - x_n| < 1/n$ and $|f(y_n) - f(x_n)| \geq 2\epsilon$. Let $x = \overline{\lim}_{n\to\infty} x_n$, which exists because the sequence of x_n is bounded. Since f is continuous at x, there is a response $\delta > 0$ to ϵ at x: $|y - x| < \delta$ implies that $|f(y) - f(x)| < \epsilon$. Because x is the upper limit of the x_n, we can find an $n > 2/\delta$ for which $|x - x_n| < \delta/2$. It follows that

$$|y_n - x| \leq |y_n - x_n| + |x_n - x| < \frac{1}{n} + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

but

$$|f(y_n) - f(x)| \geq |f(y_n) - f(x_n)| - |f(x) - f(x_n)| > 2\epsilon - \epsilon = \epsilon.$$

Q.E.D.

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