Derivation of Fourier's Solution

Appendix to *A Radical Approach to Real Analysis* 2nd edition

December 13, 2005

Fourier began his study of the heat flow problem by demonstrating that a stationary solution satisfies the differential equation now known as Laplace's equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial w^2} = 0. \tag{1}$$

Pierre Simon Laplace (1749–1827) and others had come across this equation in various contexts. In modern terminology, it is simply the observation that when the flow of heat (∇z) has reached a state of equilibrium, it is incompressible ($\nabla \cdot \nabla z = 0$).

Web Resource: Go to **Laplace's Equation** to learn more about this fundamental partial differential equation.

To solve his partial differential equation (1), Fourier introduced a technique that is standard today. He searched for special solutions of the form

$$z(x, w) = g(x)h(w). \tag{2}$$

When z is of this form, equation (1) reduces to

$$g''(x)h(w) + g(x)h''(w) = 0, \tag{3}$$

or, assuming the second derivatives are not zero,

$$\begin{aligned} \frac{h(w)}{h''(w)} + \frac{g(x)}{g''(x)} &= 0, \\ \frac{g(x)}{g''(x)} &= \frac{-h(w)}{h''(w)}. \end{aligned} \tag{4}$$

The left side of equation (4) is independent of w while the right side is independent of x. This implies that both sides are independent of both w and x, and so each of these ratios is constant,

$$\frac{g(x)}{g''(x)} = C = \frac{-h(w)}{h''(w)}.$$

Since $g(x) = Cg''(x)$, the sign of $g(x)$ is either always the same as the sign of $g''(x)$, or it is always the opposite. If we want $z(x,w)$ to be continuous, then we need to have $g(-1) = g(1) = 0$, and so $g(x)/g''(x)$ must be negative:

$$\frac{h(w)}{h''(w)} = A > 0, \qquad \frac{g(x)}{g''(x)} = -A < 0,$$

for some positive constant A. Fourier set $A = 1/t^2$ and solved for $g(x) = c_1 \cos tx + c_2 \sin tx$ and $h(w) = c_3 e^{-tw} + c_4 e^{tw}$. The coefficient of $\sin tx$ must be zero because g is an even function of x. He then argued that c_4 must be zero because the temperature will approach 0 as we move away from the source of heat at $w = 0$. He had found a solution:

$$z(x,w) = ae^{-tw} \cos tx,$$

where a and t are unknown constants. If we want this solutionto be zero at $x = \pm 1$, then t must be an odd multiple of $\pi/2$.

The general solution is a sum of such functions:

$$\begin{aligned} z(x,w) &= a_1 e^{-\pi w/2} \cos(\pi x/2) + a_2 e^{-3\pi w/2} \cos(3\pi x/2) \\ &\quad + a_3 e^{-5\pi w/2} \cos(5\pi x/2) + \cdots + a_n e^{-(2n-1)\pi w/2} \cos\big((2n-1)\pi x/2\big). \end{aligned}$$

