Laplace's Equation

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Heat can be thought of as a fluid or a gas. In the absence of external forces, it moves from regions of high density to regions of low density in a manner that is very similar to a gas. In particular, it moves along curves perpendicular to the isoclines or curves of constant temperature. If $z(x, w)$ denotes the temperature at point (x, w), then the vector representing the flow of heat at (x, w) will be the gradient of z which is

$$\nabla z = \frac{\partial z}{\partial x}\,\vec{\imath} + \frac{\partial z}{\partial w}\,\vec{\jmath}.$$

The **divergence** of a vector function is a measure of how much more of whatever is flowing leaves a given region than enters it. Given a region R and a flow described by the vector function $\vec{F} = f\,\vec{\imath} + g\,\vec{\jmath}$, the divergence at a point is measured by calculating the net rate at which the flow leaves the region R,

$$\oint_{\partial R} \vec{F} \cdot \vec{n}\, ds,$$

dividing by the area of R, and then taking the limit as the region R shrinks to the single point in question. This value is denoted by $\text{div}\vec{F} = \nabla \cdot \vec{F}$ and can be calculated directly as

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial w}.$$

As long as we are not on the boundary of our thin plate, heat is neither being created nor destroyed. Since the temperature has reached steady state (it is independent of time), the divergence must be 0. This is the same as saying that $\nabla \cdot \nabla z = 0$ or, equivalently, that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial w^2} = 0.$$

This is Laplace's equation, valid for any incompressible fluid with potential function z.