# The General Solution

Appendix to *A Radical Approach to Real Analysis* $2^{\text{nd}}$ edition


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How did Fourier discover that in order to expand $f(x) = 1$, the coefficient of $\cos((2n-1)\pi x/2)$ should be $(-1)^{n-1} \cdot 4/(2n-1)\pi$? He gave several different derivations, but they all amounted to what has become the standard procedure for finding the coefficients in a Fourier series. To keep life simple, we will restrict our attention to even functions, $f(x) = f(-x)$, because these can be expressed in terms of cosines. In chapter 6, we will look at the case of Fourier series for more general functions.

We begin with the assumption that our function actually can be written as cosine series, though it may require infinitely many terms. We begin with the equation

$$f(x) = a_1 \cos\frac{\pi x}{2} + a_2 \cos\frac{3\pi x}{2} + a_3 \cos\frac{5\pi x}{2} + \cdots = \sum_{m=1}^{\infty} a_m \cos\left(\frac{(2m-1)\pi x}{2}\right), \tag{1}$$

where the coefficients $a_1, a_2, a_3, \ldots$ exist, we just do not know what they are.

There is a nice trick for finding these coefficients. We observe that

$$\int_{-1}^{1} \cos\left(\frac{(2m-1)\pi x}{2}\right) \cdot \cos\left(\frac{(2n-1)\pi x}{2}\right) dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases} \tag{2}$$

**Web Resource:** Go to **The Orthogonality Relation** to to see why this is true.

Fourier now uses equation (2) to peel off the coefficients one at a time:

$$\begin{aligned}
\int_{-1}^{1} f(x)\cos\left(\frac{(2n-1)\pi x}{2}\right)dx &= \int_{-1}^{1}\left[\sum_{m=1}^{\infty} a_m \cos\left(\frac{(2m-1)\pi x}{2}\right)\right]\cos\left(\frac{(2n-1)\pi x}{2}\right)dx \\
&= \sum_{m=1}^{\infty} a_m \int_{-1}^{1}\cos\left(\frac{(2m-1)\pi x}{2}\right)\cdot\cos\left(\frac{(2n-1)\pi x}{2}\right)dx \\
&= a_1\cdot 0 + a_2\cdot 0 + \cdots + a_{n-1}\cdot 0 + a_n\cdot 1 + a_{n+1}\cdot 0 + \cdots \\
&= a_n. \qquad (3)
\end{aligned}$$

It is now possible to calculate the coefficients for the solution when $f(x) = 1$. The coefficients are found by substituting 1 for $f(x)$ in equation (3):

$$\begin{aligned}
a_n &= \int_{-1}^{1} 1\cdot\cos\left(\frac{(2n-1)\pi x}{2}\right)dx \\
&= \frac{2}{(2n-1)\pi}\sin\left(\frac{(2n-1)\pi x}{2}\right)\Bigg|_{x=-1}^{x=1} \\
&= \frac{4}{(2n-1)\pi}(-1)^{n-1}. \qquad (4)
\end{aligned}$$

When $1 < x < 1$, we have

$$\begin{aligned}
1 &= \frac{4}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{2}\right) + \cdots\right] \\
&= \frac{4}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\left(\frac{(2n-1)\pi x}{2}\right). \qquad (5)
\end{aligned}$$

There is one particularly nice consequence of equation (5). If we set $x = 0$, then all the cosines take on the value 1. This implies that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots. \qquad (6)$$

**Web Resource:** Go to **Approximating Fourier's Solution** to approximate this solution and to explore the Fourier cosine series for other functions.