The Orthogonality Relation

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The term **orthogonal** means "at right angles." The concept comes from geometry, but it is not too much of a stretch to see how it comes to be applied to functions.

An easy way to determine whether or not two vectors are orthogonal is to take their inner product (also called the dot product). The inner product has two equivalent definitions. On the one hand, it is the product of the lengths of the vectors multiplied by the cosine of the angle between them,

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta.$$

On the other hand, if we know the decomposition of these vectors into the unit basis vectors, then the dot product is the sum of the products of the corresponding coefficients,

$$\left(v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}\right) \cdot \left(w_1\vec{\imath} + w_2\vec{\jmath} + w_3\vec{k}\right) = v_1w_1 + v_2w_2 + v_3w_3.$$

The first definition gives meaning to the the inner product. The second makes it easy to calculate. From the first definition, we can use the inner product to decide whether or not two vectors are orthogonal: They are orthogonal if and only if their inner product is 0. We can also use it to find the **norm** or length of any vector. Since the angle between any vector and itself is 0, we have that

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.$$

Functions are like vectors in that the sum of two functions is another function, and any constant multiple of a function is a function. If we can define a natural inner product on functions, then we can use it to find the norm of a function (analogous to the length of a vector), and we can use it to define orthonality of functions.

A natural inner product for functions is given by the integral of their ordinary product. This is the limit of a sum of products, and so really is analogous to the inner product of vectors. We denote this inner product by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

It follows that the **norm** or “length” of a function is

$$\|f\| = \sqrt{\langle f, f\rangle} = \left(\int_{-1}^{1} f^2(x)\,dx\right)^{1/2}.$$

Two functions are **orthogonal** if and only if

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx = 0.$$

The functions

$$\{\cos(\pi x/2), \cos(3\pi x/2), \cos(5\pi x/2), \cos(7\pi x/2), \ldots\}$$

are pairwise orthogonal—any two distinct functions from this list are orthogonal—and their norms are all 1:

$$\begin{aligned}\left\langle \cos\left(\frac{(2m-1)\pi x}{2}\right), \cos\left(\frac{(2n-1)\pi x}{2}\right)\right\rangle \\ &= \int_{-1}^{1} \cos\left(\frac{(2m-1)\pi x}{2}\right)\cos\left(\frac{(2n-1)\pi x}{2}\right)dx \\ &= \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n. \end{cases} \qquad (1)\end{aligned}$$

To prove equation (1), we start with the trigonometric identity,

$$\cos(A)\cos(B) = \frac{1}{2}\big(\cos(A+B) + \cos(A-B)\big).$$

When we use this to simplify our integral, we get that

$$\begin{aligned}\int_{-1}^{1} \cos\left(\frac{(2m-1)\pi x}{2}\right)\cos\left(\frac{(2n-1)\pi x}{2}\right)dx \\ &= \frac{1}{2}\int_{-1}^{1}\big[\cos\big((m+n-1)\pi x\big) + \cos\big((m-n)\pi x\big)\big]\,dx.\end{aligned}$$

If $m = n$, this is

$$\frac{1}{2}\int_{-1}^{1}\big[\cos\big((2n+1)\pi x\big) + 1\big]\,dx = \frac{1}{2}\left[\frac{\sin\big((2n+1)\pi x\big)}{(2n-1)\pi} + x\right]_{-1}^{1} = 1.$$

If $m \neq n$, this is

$$\frac{1}{2}\left[\frac{\sin\big((m+n-1)\pi x\big)}{(m+n-1)\pi} + \frac{\sin\big((m-n)\pi x\big)}{(m-n)\pi}\right]_{-1}^{1} = 0.$$

