

# Fourier Series as Complex Power Series

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As we shall prove in chapter 5, power series are nice. You can integrate a function represented by its power series by integrating term-by-term. The resulting series will converge to an integral of the original function at every point in the interval of convergence of the original function. You can also differentiate a power series by differentiating term-by-term, but here you have to be a little more careful. The infinite sum of derivatives will have the same radius of convergence as the original summation, but the interval of convergence could be strictly smaller. For example,

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \quad (1)$$

has  $[1, 1)$  as its interval of convergence, converging at  $x = 1$  but not at  $x = 1$ . We can differentiate term-by-term,

$$f'(x) = 1 + x + x^2 + x^3 + \dots, \quad (2)$$

which has interval of convergence  $(-1, 1)$ , converging at neither  $x = -1$  nor at  $x = 1$ .

In chapter 5, we shall prove that when we integrate or differentiate a power series, we do not change the radius of convergence, but we can lose points of convergence on the boundary when we differentiate. When integrating, we can gain points of convergence on the boundary.

The key to understanding Fourier series is that they are really power series in the complex plane, using the relationship

$$e^{ix} = \cos x + i \sin x, \quad (e^{ix})^n = e^{inx} = \cos nx + i \sin nx. \quad (3)$$

The series

$$f(x) = \frac{4}{\pi} \left[ \cos(\pi x/2) - \frac{1}{3} \cos(3\pi x/2) + \frac{1}{5} \cos(5\pi x/2) - \frac{1}{7} \cos(7\pi x/2) + \dots \right] \quad (4)$$

is simply the real part of

$$\frac{4}{\pi} \left[ z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \dots \right], \quad (5)$$

where  $z = e^{i\pi x/2}$ .

The series in (5) has radius of convergence equal to 1.

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Given any power series in  $z$  with radius of convergence  $R$ , the series will converge when  $|z| < R$  and diverge when  $|z| > R$ . In other words, it converges at all points inside the circle of radius  $R$  (which is why we call it the radius of convergence). Behavior on the circle that forms the boundary between the region of convergence and the region of divergence is more complicated. The series might converge at all points on the circle, or it might diverge at all points on the circle, or it could converge at some points and diverge at others.

The Fourier series is a power series in  $z$  evaluated at  $z = e^{(i\pi x/2)}$ . All of these points lie on the circle centered at the origin with radius 1. For the power series in (5), these are all points on the boundary between the region of convergence and the region of divergence. This is why these series are so problematic. It explains why Fourier series can converge everywhere, yet the term-by-term derivative fails to converge anywhere except where all summands are 0.