

Newton's Formula

Appendix to *A Radical Approach to Real Analysis* 2nd edition
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We know how Newton discovered his binomial theorem because he described the process in a letter to Leibniz.

Following in Wallis's footsteps, Newton recognized that the key to calculating π was finding a way of evaluating $\pi/4 = \int_0^1 (1-x^2)^{1/2} dx$. If that exponent were an integer instead of $1/2$, life would be easy. Like Wallis, Newton begins by comparing what he has to what he can evaluate. He looks at the expansions of $(1+x)^m$ for integer values of m .

$$\begin{aligned}(1+x)^0 &= 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots, \\(1+x)^1 &= 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots, \\(1+x)^2 &= 1 + 2 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots, \\(1+x)^3 &= 1 + 3 \cdot x + 3 \cdot x^2 + 1 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots, \\(1+x)^4 &= 1 + 4 \cdot x + 6 \cdot x^2 + 4 \cdot x^3 + 1 \cdot x^4 + 0 \cdot x^5 + \cdots, \\&\vdots\end{aligned}$$

He considered a table of the coefficients and tried to guess what coefficients would correspond to an exponent of $m = 1/2$ in

$$(1+x)^m = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots.$$

m	x^0	x^1	x^2	x^3	x^4	x^5
0	1	0	0	0	0	0
1/2						
1	1	1	0	0	0	0
3/2						
2	1	2	1	0	0	0
5/2						
3	1	3	3	1	0	0
7/2						
4	1	4	6	4	1	0
9/2						
5	1	5	10	10	5	1

It is easy to guess that the values in the first column are all 1, and the values in the second column must equal m . What about the third column?

Newton would have been very familiar with the sequence 1, 3, 6, 10, , the **triangular numbers**. The j th triangular number is the sum of the integers from 1 to j . It equals $j(j+1)/2$. The exponent m corresponds to the $(m - 1)$ st triangular number, so the formula to use in the third column is $m(m - 1)/2$.

If the values in the first column are constant, the values in the second column increase linearly, and the values in the third column are given by a quadratic formula, then it makes sense to look for a cubic polynomial for the fourth column, a quartic polynomial for the fifth, and so on. Armed with this assumption, it is not difficult to determine what these polynomials must be.

We know that the cubic polynomial in m that fits the coefficients of x^3 must have roots at $m = 0$, 1, and 2. This cubic polynomial must be $cm(m - 1)(m - 2)$ for some still to be determined constant c . We can find c by using the fact that that this polynomial is 1 when $m = 3$:

$$1 = c \cdot 3(3 - 1)(3 - 2) = 6c.$$

This polynomial is $m(m - 1)(m - 2)/6$.

A similar argument shows us that the polynomial in the next column should be $m(m - 1)(m - 2)(m - 3)/4!$ and the polynomial in the fifth column should be $m(m - 1)(m - 2)(m - 3)(m - 4)/5!$.

In general, the column that corresponds to x^k will have zeros at $m = 0, 1, 2, \dots, k - 1$, and a 1 at $m = k$. The corresponding polynomial is

$$\frac{m(m - 1)(m - 2) \cdots (m - k + 1)}{k!}.$$

Since this is defined for any value of m , we can fill in the table:

m	x^0	x^1	x^2	x^3	x^4	x^5
0	1	0	0	0	0	0
1/2	1	1/2	-1/8	1/16	-5/128	7/256
1	1	1	0	0	0	0
3/2	1	3/2	3/8	-1/16	3/128	-3/256
2	1	2	1	0	0	0
5/2	1	5/2	15/8	5/16	-5/128	3/256
3	1	3	3	1	0	0
7/2	1	7/2	35/8	35/16	35/128	-7/256
4	1	4	6	4	1	0
9/2	1	9/2	63/8	105/16	315/128	63/256
5	1	5	10	10	5	1

All of this has been inspired guesswork. Newton did not supply a proof at this time, but he recognized that this enabled him to calculate π with great accuracy, and therefore he was certain

that it must be correct. He had discovered the general binomial theorem:

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \dots + \frac{m(m-1)\cdots(m-k+1)}{k!} x^k + \dots . \quad (1)$$