

How to find and write a proof

Appendix to *A Radical Approach to Real Analysis* 2nd edition
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March 31, 2006

This document lays out the steps of finding a proof of an if-then statement. We will use as an example exercise 14 of section 3.1: “Prove that if f is continuous at a and $\lim_{x \rightarrow a} f'(x)$ exists, then so does $f'(a)$, and they must be equal.” In section 2, we give some further ideas to help when stymied in step 4.

1 Five steps

Step 1: Identify the hypothesis and the conclusion. Usually the hypothesis is signaled by the word “if” and the conclusion by “then”, but there are other ways of expressing a theorem. For example, we might have written, “Prove that $f'(a)$ exists whenever f is continuous at a and $\lim_{x \rightarrow a} f'(x)$ exists, and show that when this happens they must be equal.”

In either case, the hypothesis and conclusion are the same:

Hypothesis: f is continuous at a AND $\lim_{x \rightarrow a} f'(x)$ exists

Conclusion: $f'(a)$ exists AND $f'(a) = \lim_{x \rightarrow a} f'(x)$

Note that there are two statements in the conclusion, each of which will need to be proven.

Step 2: Focus first on the conclusion. What does it actually say? What statements can you think of that would immediately imply this conclusion? How are the terms in this conclusion defined?

Start with the statement “ $f'(a)$ exists.” The Cauchy explanation of this statement is that there is a number—which we shall call $f'(a)$ —with the property that if we define

$$E(x, a) = f'(a) - \frac{f(x) - f(a)}{x - a},$$

then we can force $|E(x, a)|$ to be as small as we wish simply by taking x sufficiently close to a . If we can show that, given any $\epsilon > 0$, there is such a number for which we can find a response $\delta > 0$ so that $|E(x, a)| < \epsilon$ for all x satisfying $0 < |x - a| < \delta$, then we will have the first conclusion.

The second part of the conclusion says that $f'(a)$ is equal to $\lim_{x \rightarrow a} f'(x)$. This tells us how to find our candidate for $f'(a)$. We need to show that given an $\epsilon > 0$, there is a response $\delta > 0$ so that

$$\left| \lim_{x \rightarrow a} f'(x) - \frac{f(x) - f(a)}{x - a} \right| < \epsilon \quad (1)$$

for all x satisfying $0 < |x - a| < \delta$. If we can prove this, then we have demonstrated the conclusion. This is where we want to head.

Step 3: Now look at the hypothesis. What does it say that might help get to our reformulation of the conclusion? How are the terms of the hypothesis defined?

For our example, the hypothesis states that $\lim_{x \rightarrow a} f'(x)$ exists. This means that there is a target value T and we can force $f'(x)$ to be as close as we wish to T by taking x sufficiently close to a . The hypothesis that f is continuous at a is clearly important. You should be able to think of an example of a function that is not continuous at a and for which $\lim_{x \rightarrow a} f'(x)$ exists, but $f'(a)$ does not exist. But it is not yet clear how we will use it.

Step 4: Now comes the hard part. We begin a process of comparing what we know from the hypothesis with what we want to show in order to arrive at the conclusion. Is there some result that follows from the hypothesis that will get us closer to the conclusion? Is there another statement that implies the conclusion that looks a little more like what we know from the hypothesis? We work from both ends trying to bring them closer. There is no guaranteed route to success. You keep trying ideas until you find something that works. Think of it as building a bridge, working out across a gorge from each side until they link up.

There is one quick simplification we can make now that we have assigned T as the target value of the limit of $f'(x)$. If we can show that for any $\epsilon > 0$ there is a response $\delta > 0$ so that

$$\left| T - \frac{f(x) - f(a)}{x - a} \right| < \epsilon \quad (2)$$

for all x satisfying $0 < |x - a| < \delta$, then we have finished the proof.

In addition to trying to bring the hypothesis and conclusion as close together as possible, we also scour the results we know to see if anything might be relevant. In this case, we do not yet know many theorems. There is one that might help: the mean value theorem (theorem 3.1). It would enable us to replace $(f(x) - f(a)) / (x - a)$ with $f'(c)$. Before we try to use it, check that the hypotheses of this theorem are satisfied:

1. Is f differentiable at all points strictly between x and a ? We do not know that it is differentiable at a , but we do not need to know that. We know that $\lim_{x \rightarrow a} f'(x)$ exists, and so f' must exist for all values sufficiently close to a . The first hypothesis is satisfied.
2. Is f continuous at every point on the closed interval from a to x ? Differentiability implies continuity. The only problem that we might have is continuity at a . Here is where we use the other part of our hypothesis. We were told that f is continuous at a .

We can use the mean value theorem to find a simpler form of equation (2). We know that there is a c strictly between a and x for which $(f(x) - f(a)) / (x - a) = f'(c)$. If x is within δ of a , then c will also be within δ of a . Given $\epsilon > 0$, we need to show that there is a response $\delta > 0$ so that if c is within δ of a , then

$$|T - f'(c)| < \epsilon \quad (3)$$

But that is just the definition of this target value. We have found a proof!

Step 5: This is the step that too many students skip, but you have not given a proof if you stop after completing Step 4. The hard work is done. You have connected the two ends, the hypotheses and the conclusions. Now you need to write up the proof. You lay out for your readers a logical progression that takes them directly from the hypotheses to the conclusions in as painless and clear a manner as possible. You want to take the readers across your bridge in a seamless trip. Here is one example of how to rewrite the proof that has just been discovered. Notice that it includes making explicit those δ s that imply that our error function is bounded by ϵ .

Let $T = \lim_{x \rightarrow a} f'(x)$. We need to show that given any $\epsilon > 0$, there is a response δ so that $0 < |x - a| < \delta$ implies that

$$\left| T - \frac{f(x) - f(a)}{x - a} \right| < \epsilon.$$

By the definition of the limit, there is a δ so that $0 < |x - a| < \delta$ implies that $|T - f'(x)| < \epsilon$. By the mean value theorem, there is a c strictly between a and x for which

$$\frac{f(x) - f(a)}{x - a} = f'(c).$$

Since $0 < |c - a| < |x - a| < \delta$, we have that

$$\left| T - \frac{f(x) - f(a)}{x - a} \right| = |T - f'(c)| < \epsilon.$$

Some of the details have been left out. When they need to be included is a matter of judgment. Details often obscure the essence of the proof, but if you think that your readers would stop and puzzle over certain points, then you need to put in those details. For example, if your readers are likely to wonder why you can use the mean value theorem, then you should include your analyses of the hypotheses of that theorem. Also notice that I have chosen to start the proof with the observation made in Step 2. Again, this is for the benefit of the readers, to help them see where I am going with this proof. Sometimes it helps to begin the proof by stating what needs to be done to reach the conclusion. Sometimes this is not necessary. It might even be confusing.

2 Help with Step 4

In our example, we were able to construct our bridge by working out from the hypotheses and conclusions until we found a link. That is often very hard. There are two other variations that can be helpful.

Using the contra-positive

Instead of trying to prove that if A , then B , try to prove the **contra-positive**, the logically equivalent statement that if B is false then A is false. You need to be particularly careful when taking the negation of a statement that contains a conjunction: “and” or “or”. The negation of “ A and B ” is “not A or not B ”. The contra-positive of our example is:

If $f'(a)$ does not exist or it exists but does not equal $\lim_{x \rightarrow a} f'(x)$, then $\lim_{x \rightarrow a} f'(x)$ does not exist or f is not continuous at a .

Taking the contra-positive does not make it any easier to tackle this particular theorem, but it can be helpful. You are now trying to build your bridge in a different location.

Search for a contradiction

One of the most powerful tools for bridging the gap is to assume that the hypothesis is true and the conclusion is false. If you can show that this leads to an impossible situation, then whenever the hypothesis is true, the conclusion must be true, and so you have proven your theorem. For our example, we would assume that

1. f is continuous at a ,
2. $\lim_{x \rightarrow a} f'(x)$ exists, and
3. $f'(a)$ does not exist or it exists but does not equal $\lim_{x \rightarrow a} f'(x)$

You now explore what these assumptions tell you about the function, looking for some contradiction.

The classic example of the use of proof by contradiction is the standard proof that $x^2 = 2$, then x is irrational. the statements with which we get to work are

1. $x^2 = 2$,
2. x is rational.

We look for consequences of these statements that produce a contradiction. Since x is rational, we can write it as $x = m/n$ where m and n are relatively prime integers. Since $x^2 = 2$, we can substitute m/n for x and clear denominators,

$$\left(\frac{m}{n}\right)^2 = 2 \implies m^2 = 2n^2.$$

The right side of this equality is an even integer, so m^2 is even and that means that m is even. We can find an integer t such that $m = 2t$. We substitute $2t$ for m in our last equation,

$$(2t)^2 = 2n^2 \implies 2t^2 = n^2.$$

Now we see that n must be even. We have our contradiction because m and n are both even but they are also relatively prime.