Continued Fractions

Appendix to *A Radical Approach to Real Analysis* 2^{nd} edition

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To get out sequence that approaches π, we begin by splitting off the great integer less than or equal to π,
$floor \pi \rfloor = a_1 = 3$,

$$\pi = 3 + r_1, \quad r_1 = 0.14159\ldots\ .$$

The remainder, r_1, lies between 0 and 1, so its reciprocal is larger than 1:

$$\frac{1}{r_1} = 7.0625133\ldots\ .$$

Again, we split off the greatest integer, $a_2 = 7$, and consider the new remainder $r_2 = 0.0625133\ldots$. We keep doing this, generating a sequence of positive integers:

$$a_1 = 3,\ a_2 = 7,\ a_3 = 15,\ a_4 = 1,\ a_5 = 292,\ a_6 = 1,\ a_7 = 1,\ a_8 = 1,\ a_9 = 2,\ \ldots\ .$$

If we stop after the kth integer, we get a rational approximation to π,

$$\begin{aligned}
\frac{p_1}{q_1} &= 3 = \frac{3}{1}, \\
\frac{p_2}{q_2} &= 3 + \frac{1}{7} = \frac{22}{7}, \\
\frac{p_3}{q_3} &= 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106}, \\
\frac{p_3}{q_3} &= 3 + \frac{1}{7 + \frac{1}{15 + 1/1}} = \frac{355}{113}, \\
&\vdots
\end{aligned}$$

This is called a continued fraction. There is a special notation that makes it easier to write long continued fractions:

$$\frac{p_9}{q_9} = 3 + \frac{1}{7+}\ \frac{1}{15+}\ \frac{1}{1+}\ \frac{1}{292+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2} = \frac{833719}{265381}.$$

These give the best possible rational approximations for a given limit on the denominator. Specifically, we shall see that if $1 \leq b < q_k$, then no fraction with denominator b can be closer to π than p_k/q_k.

There is nothing special about π in all of this. We could have started with any other irrational number such as $\sqrt{2}$ or e. In fact, while the sequence for π has no discernible pattern, there are very simple patterns for the integers in the sequences for $\sqrt{2}$ and e. As long as we start we with an irrational number, the sequence will never end. The sequence terminates if and only if we start with a rational number. In what follows, we shall prove everything for an arbitrary irrational number that we call α.

We begin by defining the sequence:

$$\begin{aligned} a_1 &= \lfloor \alpha \rfloor, \\ r_1 &= \alpha - a_1, \\ a_{k+1} &= \left\lfloor \frac{1}{r_k} \right\rfloor, \\ r_{k+1} &= \frac{1}{r_k} - a_{k+1}. \end{aligned}$$

We also define the sequence of rational approximations,

$$\frac{p_k}{q_k} = a_1 + \frac{1}{a_2+}\ \frac{1}{a_3 + \cdots}\ \ \frac{1}{a_k}.$$

Notice that if we replace the last a_k by $a_k + r_k$ (an irrational number), we get an expression that exactly equals our original number,

$$\alpha = a_1 + \frac{1}{a_2+}\ \frac{1}{a_3 + \cdots}\ \ \frac{1}{a_k + r_k}.$$

Proposition 1. *If we define* $p_0 = 1$, $p_1 = a_1$, $q_0 = 0$, $q_1 = 1$, *then for all* $k \geq 1$, *we can define*

$$\begin{aligned} p_{k+1} &= p_{k-1} + a_{k+1} p_k, && (1) \\ q_{k+1} &= q_{k-1} + a_{k+1} q_k. && (2) \end{aligned}$$

Furthermore, we have that

$$p_{k+1} q_k - p_k q_{k+1} = (-1)^{k+1}, \tag{3}$$

and, therefore, $\gcd(p_k, q_k) = 1$ *for all* $k \geq 0$.

Proof. We prove these equations by induction. When $k = 1$, we have that

$$p_2 = a_1 a_2 + 1 = p_0 + a_2 p_1, \quad q_2 = a_2 = q_0 + a_2 q_1, \quad p_2 q_1 - p_1 q_2 = a_1 a_2 + 1 - a_1 a_2 = 1.$$

We can turn p_k/q_k into p_{k+1}/q_{k+1} by taking the continued fraction for p_k/q_k and replacing a_k with $a_k + 1/a_{k+1}$. By our induction hypothesis,

$$p_k = p_{k-2} + a_k p_{k-1}, \quad q_k = q_{k-2} + a_k q_{k-1}.$$

Therefore,

$$\begin{aligned}
\frac{p_{k+1}}{q_{k+1}} &= \frac{p_{k-2} + (a_k + 1/a_{k+1})p_{k-1}}{q_{k-2} + (a_k + 1/a_{k+1})q_{k-1}} \\
&= \frac{p_{k-2} + a_k p_{k-1} + (1/a_{k+1})p_{k-1}}{q_{k-2} + a_k q_{k-1} + (1/a_{k+1})q_{k-1}} \\
&= \frac{p_k + p_{k-1}/a_{k+1}}{p_k + p_{k-1}/a_{k+1}} \\
&= \frac{p_{k-1} + a_{k+1}p_k}{q_{k-1} + a_{k+1}q_k}.
\end{aligned}$$

By our induction hypothesis, $p_k q_{k-1} - p_{k-1} q_k = (-1)^k$. Using equations (1) and (2), we have that

$$\begin{aligned}
p_{k+1}q_k - p_k q_{k+1} &= (p_{k-1} + a_{k+1}p_k)q_k - p_k(q_{k-1} + a_{k+1}q_k) \\
&= p_{k-1}q_k + a_{k+1}p_k q_k - p_k q_{k-1} - a_{k+1}p_k q_k \\
&= p_{k-1}q_k - p_k q_{k-1} = -(-1)^k.
\end{aligned}$$

By equation (3), any common divisor of p_k and q_k is a divisor of 1. □

Because we also round down to the next integer, p_k/q_k is less than α when k is odd, greater than α when k is even. The difference between two successive approximations is

$$\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{p_{k+1}q_k - p_k q_{k+1}}{q_k q_{k+1}} = \frac{(-1)^{k+1}}{q_k q_{k+1}}. \tag{4}$$

This tells us that we can write p_k/q_k as the sum of an alternating series,

$$\frac{p_k}{q_k} = a_1 + \sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{q_j q_{j+1}}.$$

Since the values of q_k increase, the summands are decreasing and approach 0. This is an alternating series that converges to α. The partial sums alternate larger and smaller than α.

If a/b lies between p_k/q_k and p_{k+1}/q_{k+1}, then

$$\frac{1}{q_k q_{k+1}} > \left| \frac{a}{b} - \frac{p_k}{q_k} \right| = \frac{|aq_k - bp_k|}{bq_k} \geq \frac{1}{bq_k},$$

and therefore $b > q_{k+1}$.