Exponential Function

Appendix to *A Radical Approach to Real Analysis* 2^{nd} edition

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The associated *Maple* and *Mathematica* notebooks will let you explore the convergence of $\left(1+x/n\right)^n$. Using the fact that the natural logarithm is a continuous function and therefore the natural logarithm of this limit is the same as the limit of the natural logarithm, we have that

$$\begin{aligned}
\ln\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) &= \lim_{n\to\infty}\ln\left(\left(1+\frac{x}{n}\right)^n\right) \\
&= \lim_{n\to\infty} n\,\ln\left(1+\frac{x}{n}\right) \\
&= \lim_{n\to\infty}\frac{\ln(1+x/n)}{n^{-1}} \\
&= \lim_{n\to\infty}\frac{-xn^{-2}/(1+x/n)}{-n^{-2}} \\
&= \lim_{n\to\infty}\frac{x}{1+x/n} \\
&= x.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

Another way to approach this identity is to use the binomial expansion:

$$\begin{aligned}
\left(1+\frac{x}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k} \\
&= \sum_{k=0}^{n}\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!\;n^k}\,x^k \\
&= \sum_{k=0}^{n}\frac{1(1-1/n)(1-2/n)\cdots(1-(k-1)/n)}{k!}\,x^k.
\end{aligned}$$

As n approaches infinity, the number of summands becomes infinite and the kth summand becomes $x^k/k!$:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x.$$