Convergence in Norm

Appendix to *A Radical Approach to Real Analysis* 2^{nd} edition

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In *A Radical approach to Real Analysis*, we only considered pointwise convergence of a sequence of functions, but in *a Radical Approach to Lebesgue's Theory of Integration*, other types of convergence will be considered including convergence in norm. We define the distance between two functions, f and g, over an interval, say $[0,1]$, by

$$\|f-g\| = \int_0^1 \left|f(x)-g(x)\right| dx.$$

The distance between two functions is 0 if and only if the integral of the absolute value of difference between these functions is 0. Given two continuous functions, this can happen only if they are equal everywhere (see exercise 1).

We say that the sequence $(f_n)_{n=1}^{\infty}$ converges to f if for each $\epsilon > 0$ there is a response N so that $n \geq N$ implies that

$$\|f_n - f\| < \epsilon.$$

Exercises

1. Prove that if f and g are continuous on $[0,1]$, then $\|f-g\| = 0$ if and only if $f(x) = g(x)$ for all $x \in [0,1]$.

2. Define the sequence of functions

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n, \quad 0 \leq x \leq 1, \quad n \geq 1.$$

Show that f_n converges to e^x both pointwise and in norm over $[0,1]$.

3. Define the sequence of functions

$$g_n(x) = nxe^{-nx^2}, \quad 0 \leq x \leq 1, \ n \geq 1.$$

Show that g_n converges pointwise to the constant function $g(x) = 0$. Show that it does not converge in norm to *any* function.

4. Consider the functions $f_{n,k}$ defined for $0 \le x \le 1$, $1 \le k \le n$, by

$$f_{n,k}(x) = \begin{cases} 1, & \text{if } (k-1)/n \le x \le k/n, \\ 0, & \text{otherwise.} \end{cases}$$

We create a sequence,

$$f_{1,1}, f_{2,1}, f_{2,2}, f_{3,1}, f_{3,2}, f_{3,3}, f_{4,1}, f_{4,2}, \ldots.$$

Show that this sequence does not converge pointwise at any point in $[0, 1]$, but it does converge in norm to the constant function $f(x) = 0$