Gauss's Test

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Theorem 1. Gauss's Test *Let*

$$f(x) = x^\beta(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots), \quad a_i \neq 0,$$

be a hypergeometric power series. We assume that the polynomials in the numerator and denominator of a_{n+1}/a_n have the same degree:

$$\frac{a_{n+1}}{a_n} = \frac{C_t n^t + C_{t-1} n^{t-1} + \cdots + C_0}{c_t n^t + c_{t-1} n^{t-1} + \cdots + c_0},$$

where neither C_t nor c_t is zero. The radius of convergence is $|c_t/C_t|$. If $|x| = |c_t/C_t|$, then we can write the absolute value of the ratio of successive terms as

$$\left|\frac{a_{n+1}}{a_n}\, x\right| = \left|\frac{n^t + B_{t-1} n^{t-1} + \cdots + B_0}{n^t + b_{t-1} n^{t-1} + \cdots + b_0}\right|, \tag{1}$$

where $B_j = C_j/C_t$ and $b_j = c_j/c_t$. The test is as follows:

1. *If $B_{t-1} > b_{t-1}$, then the absolute values of the summands grow without limit and the series cannot converge.*
2. *If $B_{t-1} = b_{t-1}$, then the absolute values of the summands approach a finite nonzero limit and the series cannot converge.*
3. *If $B_{t-1} < b_{t-1}$, then the absolute values of the summands approach zero. If the series is alternating, then it converges.*
4. *If $B_{t-1} \geq b_{t-1} - 1$, then the series is not absolutely convergent.*
5. *If $B_{t-1} < b_{t-1} - 1$, then the series is absolutely convergent.*

Proof of Gauss's Test: Part I

We shall follow Gauss's proof, taking the liberty of rephrasing it and occasionally elaborating on what he is doing. The reader is encouraged to look up Gauss's original statements, portions of which have been translated into English in Garrett Birkhoff's *A Source Book in Classical Analysis*. The complete paper (in Latin) is in the third volume of *Werke*, Carl Friedrich Gauss's collected works. Following Gauss, we define $M_1 + M_2 + M_3 + \cdots$ to be our hypergeometric series,

$$\left|\frac{M_{n+1}}{M_n}\right| = \left|\frac{n^t + B_{t-1}n^{t-1} + \cdots + B_0}{n^t + b_{t-1}n^{t-1} + \cdots + b_0}\right|. \tag{2}$$

We define $P(n) = n^t + B_{t-1}n^{t-1} + \cdots + B_0$ and $p(n) = n^t + b_{t-1}n^{t-1} + \cdots + b_0$.

For large values of n, it is the coefficient of the highest power of n (the **leading coefficient**) that determines the sign of the polynomial. As Gauss points out, once you are past the rightmost root of the polynomial, then the polynomial takes on only positive values if and only if the leading coefficient is positive. It follows that once n is larger than the largest root of either $P(n)$ or $p(n)$, then

$$\left|\frac{P(n)}{p(n)}\right| = \frac{P(n)}{p(n)}. \tag{3}$$

Let k be the largest subscript for which $B_k \neq b_k$. If $B_k - b_k$ is positive, then $P(n) - p(n)$ will eventually be positive:

$$\frac{P(n)}{p(n)} = 1 + \frac{P(n) - p(n)}{p(n)} > 1, \tag{4}$$

for all n larger than the rightmost root of $p(n)[P(n) - p(n)]$. This implies that

$$\frac{|M_{n+1}|}{|M_n|} = \frac{P(n)}{p(n)} > 1$$

and so $|M_{n+1}| > |M_n|$. The $|M_n|$ are strictly increasing once we are past this rightmost root.

Similarly, if $B_k - b_k$ is negative, then $P(n) - p(n)$ will eventually be negative and

$$\frac{P(n)}{p(n)} = 1 + \frac{P(n) - p(n)}{p(n)} < 1, \tag{5}$$

for all n sufficiently large. The $|M_n|$ are strictly decreasing once n has passed the rightmost root.

Proof of Gauss's Test: Part II

Gauss's test says more than just that the absolute values of the summands are increasing or decreasing. It says that when $B_{t-1} > b_{t-1}$, they increase without limit. When $B_{t-1} < b_{t-1}$, they approach zero. In this part, we assume that $B_{t-1} \neq b_{t-1}$.

If $B_{t-1} > b_{t-1}$, then we can find an integer h such that

$$h(B_{t-1} - b_{t-1}) > 1. \tag{6}$$

We define a new series $N_1 + N_2 + N_3 + \cdots$ by

$$N_n = \frac{M_n^h}{n}. \tag{7}$$

Our new series is also hypergeometric:

$$\begin{aligned}
\left|\frac{N_{n+1}}{N_n}\right| &= \left|\frac{M_{n+1}}{M_n}\right|^h \frac{n}{n+1} \\
&= \left(\frac{P(n)}{p(n)}\right)^h \frac{n}{n+1} \\
&= \left(\frac{n^t + B_{t-1}n^{t-1} + \cdots + B_0}{n^t + b_{t-1}n^{t-1} + \cdots + b_0}\right)^h \frac{n}{n+1} \\
&= \frac{n^{th+1} + hB_{t-1}n^{th} + \cdots}{n^{th+1} + (hb_{t-1}+1)n^{th} + \cdots}. \qquad (8)
\end{aligned}$$

Since we chose h so that $hB_{t-1} > hb_{t-1} + 1$, we know from part I that the summands of our new series are also increasing in absolute value for n sufficiently large. We now observe that

$$|M_n| = \sqrt[h]{n|N_n|}. \qquad (9)$$

Since h is constant and $|N_n|$ is increasing, $|M_n|$ grows without limit as n increases.

Similarly, if $B_{t-1} < b_{t-1}$, then we can find an integer h such that

$$h(b_{t-1} - B_{t-1}) > 1. \qquad (10)$$

We define a new series $N_1 + N_2 + N_3 + \cdots$ by

$$N_n = nM_n^h. \qquad (11)$$

This series is also hypergeometric:

$$\begin{aligned}
\left|\frac{N_{n+1}}{N_n}\right| &= \left(\frac{P(n)}{p(n)}\right)^h \frac{n+1}{n} \\
&= \frac{n^{th+1} + (hB_{t-1}+1)n^{th} + \cdots}{n^{th+1} + hb_{t-1}n^{th} + \cdots}. \qquad (12)
\end{aligned}$$

Since we chose h so that $hB_{t-1} + 1 < hb_{t-1}$, we know from part I that the summands of our new series are decreasing in absolute value. We observe that

$$|M_n| = \sqrt[h]{|N_n|/n}. \qquad (13)$$

Since h is constant and $|N_n|$ is decreasing, $|M_n|$ approaches 0 as n increases.

Proof of Gauss's Test: Part III

We next tackle the case where $B_{t-1} = b_{t-1}$. We first assume that $B_k > b_k$ at the largest subscript k for which $B_k \neq b_k$. From part I, this tells us that for n sufficiently large, the absolute values of the summands will be strictly increasing.

We find an integer h with the property that

$$B_{t-2} - b_{t-2} - h < 0. \tag{14}$$

We then define a new series $N_1 + N_2 + N_3 + \cdots$ by

$$N_n = M_n \left(\frac{n}{n-1} \right)^h, \tag{15}$$

and note that

$$|N_n| > |M_n|. \tag{16}$$

Again, we have created a new hypergeometric series:

$$\begin{aligned} \left| \frac{N_{n+1}}{N_n} \right| &= \left| \frac{M_{n+1}}{M_n} \right| \left(\frac{(n+1)(n-1)}{n^2} \right)^h \\ &= \frac{P(n)}{p(n)} \left(\frac{n^2-1}{n^2} \right)^h \\ &= \frac{n^t + B_{t-1}n^{t-1} + \cdots + B_0}{n^t + b_{t-1}n^{t-1} + \cdots + b_0} \cdot \frac{n^{2h} - hn^{2h-2} + \cdots}{n^{2h}} \\ &= \frac{n^{t+2h} + B_{t-1}n^{t+2h-1} + (B_{t-2} - h)n^{t+2h-2} + \cdots}{n^{t+2h} + b_{t-1}n^{t+2h-1} + b_{t-2}n^{t+2h-2} + \cdots}. \end{aligned} \tag{17}$$

Since $B_{t-1} = b_{t-1}$ and we have chosen h so that $B_{t-2} - h < b_{t-2}$, the values of $|N_n|$ are decreasing once n is sufficiently large. We have that

$$|M_n| < |M_{n+1}| < |M_{n+2}| < \cdots < |N_{n+2}| < |N_{n+1}| < |N_n|,$$

and the distance between $|M_n|$ and $|N_n|$ approaches zero. The nested interval principle promises us that both series are approaching a common limit.

Similarly, if at the largest subscript k for which $B_k \neq b_k$ we have $B_k < b_k$, then we find an integer h for which

$$b_{t-2} - B_{t-2} - h < 0. \tag{18}$$

and define a new series by

$$N_n = M_n \left(\frac{n-1}{n} \right)^h. \tag{19}$$

We note that

$$|N_n| < |M_n|. \tag{20}$$

For this series, we have that

$$\begin{aligned} \left| \frac{N_{n+1}}{N_n} \right| &= \frac{P(n)}{p(n)} \left(\frac{n^2}{n^2-1} \right)^h \\ &= \frac{n^{t+2h} + B_{t-1}n^{t+2h-1} + B_{t-2}n^{t+2h-2} + \cdots}{n^{t+2h} + b_{t-1}n^{t+2h-1} + (b_{t-2} - h)n^{t+2h-2} + \cdots}. \end{aligned} \tag{21}$$

Since $B_{t-1} = b_{t-1}$ and we have chosen h so that $b_{t-2} - h < B_{t-2}$, the $|N_n|$ are increasing for sufficiently large values of n. We see that

$$|N_n| < |N_{n+1}| < |N_{n+2}| < \ \ldots\ < |M_{n+2}| < |M_{n+1}| < |M_n|,$$

and the distance between $|M_n|$ and $|N_n|$ approaches zero. Both series are approaching a common limit.

The remaining situation is where $P(n) = p(n)$. Here we have that $|M_{n+1}| = |M_n|$. All summands have the same absolute value.

Proof of Gauss's Test: Part IV

As we know, the fact that the summands approach zero is not enough to guarantee convergence of the series. If the series alternates in sign (or alternates in sign for all summands after some finite subscript), then we have convergence. Because the ratio of consecutive terms, M_{n+1}/M_n, is a ratio of polynomials in n, this ratio will eventually be either positive or negative and stay there. If it is negative, then our summands alternate in sign. If it is positive, then the summands have the same sign and our series converges if and only if it converges absolutely. We shall now determine when this series converges absolutely.

We first consider the case where $B_{t-1} > b_{t-1} - 1$. We observe that

$$\begin{aligned} \frac{(n+1)}{n}\left|\frac{M_{n+1}}{M_n}\right| &= \frac{(n+1)(n^t + B_{t-1}n^{t-1} + \cdots + B_0)}{n(n^t + b_{t-1}n^{t-1} + \cdots + b_0)} \\ &= \frac{n^{t+1} + (B_{t-1}+1)n^t + \cdots}{n^{t+1} + b_{t-1}n^t + \cdots}. \end{aligned} \tag{22}$$

Since $B_{t-1} + 1 > b_{t-1}$, this last fraction will eventually be larger than 1. Let m be an integer large enough so that if n is greater than or equal to m then

$$|M_{n+1}| > \frac{n}{n+1}\,|M_n|. \tag{23}$$

This implies that

$$\begin{aligned} |M_{m+k}| &> \frac{m+k-1}{m+k}\,|M_{m+k-1}| \\ &> \frac{m+k-1}{m+k}\cdot\frac{m+k-2}{m+k-1}\,|M_{m+k-2}| \\ &> \frac{m+k-1}{m+k}\cdot\frac{m+k-2}{m+k-1}\cdot\frac{m+k-3}{m+k-2}\,|M_{m+k-3}| \\ &\ \vdots \\ &> \frac{m+k-1}{m+k}\cdot\frac{m+k-2}{m+k-1}\cdots\frac{m}{m+1}\,|M_m| \\ &= \frac{m}{m+k}\,|M_m|. \end{aligned} \tag{24}$$

It now follows that

$$\begin{aligned}
|M_1| + |M_2| + |M_3| + \cdots + |M_m| + |M_{m+1}| + |M_{m+2}| + \cdots \\
&\geq |M_m| + |M_{m+1}| + |M_{m+2}| + \cdots \\
&> |M_m| + \frac{m}{m+1}|M_m| + \frac{m}{m+2}|M_m| + \frac{m}{m+3}|M_m| + \cdots \\
&= m|M_m|\left(\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3} + \cdots\right). \qquad (25)
\end{aligned}$$

This last series is the harmonic series (with a finite number of summands taken off the front end). It diverges. The comparison test tells us that $|M_1| + |M_2| + |M_3| + \cdots$ must also diverge.

If $B_{t-1} = b_{t-1} - 1$, then we find a positive integer h such that

$$B_{t-1} + B_{t-2} - b_{t-2} + h > 0. \qquad (26)$$

We observe that

$$\begin{aligned}
\frac{n+1-h}{n-h}\left|\frac{M_{n+1}}{M_n}\right| &= \frac{(n+1-h)(n^t + B_{t-1}n^{t-1} + \cdots + B_0)}{(n-h)(n^t + b_{t-1}n^{t-1} + \cdots + b_0)} \\
&= \frac{n^{t+1} + (B_{t-1} + 1 - h)n^t + (B_{t-2} + B_{t-1} - hB_{t-1})n^{t-1} + \cdots}{n^{t+1} + (b_{t-1} - h)n^t + (b_{t-2} - hb_{t-1})n^{t-1} + \cdots}. \qquad (27)
\end{aligned}$$

Since $1 = b_{t-1} - B_{t-1}$, we can rewrite inequality (26) as

$$B_{t-2} + B_{t-1} - hB_{t-1} - (b_{t-2} - hb_{t-1}) > 0. \qquad (28)$$

In the last fraction of equation (27), the coefficients of n^t are the same ($B_{t-1} + 1 - h = b_{t-1} - h$) and $B_{t-2} + B_{t-1} - hB_{t-1} > b_{t-2} - hb_{t-1}$. We choose m larger than h and large enough so that if $n \geq m$, then the fraction in equation (27) will be larger than 1. It follows that for $n \geq m$:

$$|M_{n+1}| > \frac{n-h}{n+1-h}\,|M_n|. \qquad (29)$$

This implies that

$$\begin{aligned}
|M_{m+k}| &> \frac{m+k-1-h}{m+k-h}\,|M_{m+k-1}| \\
&> \frac{m+k-1-h}{m+k-h} \cdot \frac{m+k-2-h}{m+k-1-h}\,|M_{m+k-2}| \\
&\;\vdots \\
&> \frac{m+k-1-h}{m+k-h} \cdot \frac{m+k-2-h}{m+k-1-h} \cdots \frac{m-h}{m+1-h}\,|M_m| \\
&= \frac{m-h}{m+k-h}\,|M_m|. \qquad (30)
\end{aligned}$$

It now follows that

$$\begin{aligned}
&|M_1| + |M_2| + |M_3| + \cdots + |M_m| + |M_{m+1}| + |M_{m+2}| + \cdots \\
&\quad \geq \quad |M_m| + |M_{m+1}| + |M_{m+2}| + \cdots \\
&\quad > \quad |M_m| + \frac{m-h}{m+1-h}|M_m| + \frac{m-h}{m+2-h}|M_m| + \frac{m-h}{m+3-h}|M_m| + \cdots \\
&\quad = \quad (m-h)|M_m| \left(\frac{1}{m-h} + \frac{1}{m+1-h} + \frac{1}{m+2-h} + \frac{1}{m+3-h} + \cdots \right).
\end{aligned} \tag{31}$$

Again, we can compare our series to the divergent harmonic series. The series $|M_1|+|M_2|+|M_3|+\cdots$ must diverge.

Proof of Gauss's Test: Part V

Finally, we consider the case where $B_{t-1} < b_{t-1} - 1$. We find a small positive number h such that $B_{t-1} + h$ is still less than $b_{t-1} - 1$,

$$B_{t-1} + h < b_{t-1} - 1. \tag{32}$$

We observe that

$$\begin{aligned}
\frac{n}{n-1-h} \left| \frac{M_{n+1}}{M_n} \right| &= \frac{n(n^t + B_{t-1}n^{t-1} + \cdots + B_0)}{(n-1-h)(n^t + b_{t-1}n^{t-1} + \cdots + b_0)} \\
&= \frac{n^{t+1} + B_{t-1}n^t + \cdots}{n^{t+1} + (b_{t-1} - 1 - h)n^t + \cdots}.
\end{aligned} \tag{33}$$

We know that B_{t-1} is strictly less than $b_{t-1} - 1 - h$. Eventually, this fraction will be less than and stay less than 1. We choose an integer m larger than $h+1$ and large enough so that if $n \geq m$, then

$$|M_{n+1}| < \frac{n-1-h}{n} |M_n|. \tag{34}$$

This implies that

$$\begin{aligned}
&|M_m| + |M_{m+1}| + |M_{m+2}| + \cdots \\
&\quad < \quad |M_m| + \frac{m-1-h}{m} |M_m| + \frac{(m-h)(m-1-h)}{(m+1)m} |M_m| \\
&\qquad + \frac{(m+1-h)(m-h)(m-1-h)}{(m+2)(m+1)m} |M_m| + \cdots \\
&\quad = \quad |M_m| \left(1 + \frac{m-1-h}{m} + \frac{(m-h)(m-1-h)}{(m+1)m} \right. \\
&\qquad \left. + \frac{(m+1-h)(m-h)(m-1-h)}{(m+2)(m+1)m} + \cdots \right).
\end{aligned} \tag{35}$$

We now observe that

$$
\begin{aligned}
1 &= \frac{m-1}{h} - \frac{m-1-h}{h}, \\
1 + \frac{m-1-h}{m} &= \frac{m-1}{h} - \frac{(m-1-h)(m-h)}{hm}, \\
1 + \frac{m-1-h}{m} + \frac{(m-h)(m-1-h)}{(m+1)m} & \\
&= \frac{m-1}{h} - \frac{(m-1-h)(m-h)(m-h+1)}{hm(m+1)}, \\
&\vdots \\
1 + \frac{m-1-h}{m} + \frac{(m-h)(m-1-h)}{(m+1)m} + \cdots & \\
+ \frac{(m+k-2-h)(m+k-3-h)\ldots(m-1-h)}{(m+k-1)(m+k-2)\ldots m} & \\
&= \frac{m-1}{h} - \frac{(m-1-h)(m-h)\ldots(m+k-1-h)}{hm(m+1)\ldots(m+k-1)}. \qquad (36)
\end{aligned}
$$

All of these partial sums are bounded above by $(m-1)/h$. In fact, they converge to $(m-1)/h$. Since m is larger than $h+1$,

$$1 + \frac{m-1-h}{m} + \frac{(m-h)(m-1-h)}{(m+1)m} + \cdots$$

is absolutely convergent. By comparison, the series

$$|M_m| + |M_{m+1}| + |M_{m+2}| + \cdots$$

must also converge. Since m is a fixed subscript, this series differs from the original series $|M_1| + |M_2| + |M_3| + \cdots$ by a known finite amount: $|M_1| + |M_2| + \cdots + |M_{m-1}|$. The original series converges absolutely.

□

Conclusion

This proof is an ample demonstration of the thoroughness, care, and rigor of Gauss's approach to mathematics. In fact, I have occasionally been less careful than he is in his original manuscript. If an inequality only becomes true once n passes a certain bound, then Gauss always makes explicit what this bound is and how it enters the calculations. There can be no question that Gauss fully understood the meaning of convergence and how it could be verified.

Not all power series are hypergeometric. We shall not always have clear, sharp tests for convergence. But most of the power series encountered in the real world will be hypergeometric. When the root and ratio tests return inconclusive answers, Gauss's test is the next place to turn.

And if the series is not hypergeometric ...

If the series is not hypergeometric, then Gauss's test can still be applied, but it may return an inconclusive answer.

Theorem 2. Gauss's Test for Arbitrary Series *Let* $a_1 + a_2 + a_3 + \cdots$ *be a series for which*

$$\left|\frac{a_{n+1}}{a_n}\right| = 1 + \frac{\mu + E(n)}{n},$$

where μ *is some constant and* $E(n)$ *is an error function that can be forced to be arbitrarily close to zero by taking* n *sufficiently large (given a positive error bound* ϵ*, there exists a subscript* N *such that* $n \geq N$ *implies that* $|E(n)| < \epsilon$*).*

1. *If* $\mu > 0$*, then the absolute values of the summands grow without limit and the series cannot converge.*
2. *If* $\mu = 0$ *and* $|nE(n)|$ *is bounded for all* n*, then the absolute values of the summands approach a finite nonzero limit and the series cannot converge.*
3. *If* $\mu < 0$*, then the absolute values of the summands approach zero. If the series is alternating, then it converges.*
4. *If* $\mu > -1$*, then the series is not absolutely convergent.*
5. *If* $\mu = -1$ *and* $nE(n)$ *has a lower bound (there is a number* B *and a subscript* N *such that* $n \geq N$ *implies that* $nE(n) \geq B$*), then the series is not absolutely convergent.*
6. *If* $\mu < -1$*, then the series is absolutely convergent.*

This generalization of Gauss's test enables us to handle series such as

$$1 + x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \frac{x^4}{\sqrt{4}} + \cdots.$$

The radius of convergence is

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1.$$

When $x = \pm 1$, we have that

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\sqrt{n}}{\sqrt{n+1}} \\ &= \left(1 + \frac{1}{n}\right)^{-1/2} \\ &= 1 + \frac{-1/2}{n} + \frac{(-1/2)(-3/2)}{2!\, n^2} + \cdots.\end{aligned}$$

In this case, we have $\mu = -1/2$ and

$$E(n) = \frac{(-1/2)(-3/2)}{2!\, n} + \frac{(-1/2)(-3/2)(-5/2)}{3!\, n^2} + \cdots.$$

Our series converges at $x = -1$ because it alternates in sign, but it is not absolutely convergent and so does not converge at $x = 1$.

An example of a series for which Gauss's test is inconclusive is

$$x + \frac{x^2}{2\ln 2} + \frac{x^3}{3\ln 3} + \frac{x^4}{4\ln 4} + \cdots .$$

The radius of convergence is again 1. To simplify the algebra, we shall look at the ratio $|a_n/a_{n-1}|$ rather than $|a_{n+1}/a_n|$:

$$\begin{aligned}
\left|\frac{a_n}{a_{n-1}}\right| &= \frac{(n-1)\ln(n-1)}{n\ln n} \\
&= \frac{n-1}{n}\cdot\frac{\ln(n\cdot\frac{n-1}{n})}{\ln n} \\
&= \left(1-\frac{1}{n}\right)\left(\frac{\ln n + \ln(1-n^{-1})}{\ln n}\right) \\
&= \left(1-\frac{1}{n}\right)\left(1+\frac{\ln(1-n^{-1})}{\ln n}\right) \\
&= 1 - \frac{1}{n} + \frac{\ln(1-n^{-1})}{\ln n} - \frac{\ln(1-n^{-1})}{n\ln n} \\
&= 1 - \frac{1}{n} - \left(\frac{1}{n\ln n} + \frac{1}{2n^2\ln n} + \frac{1}{3n^3\ln n} + \cdots\right) \\
&\qquad + \left(\frac{1}{n^2\ln n} + \frac{1}{2n^3\ln n} + \frac{1}{3n^4\ln n} + \cdots\right) \\
&= 1 + \frac{-1+E(n)}{n},
\end{aligned}$$

where

$$E(n) = -\frac{1}{\ln n} + \frac{1}{2n\ln n} + \frac{1}{6n^2\ln n} + \cdots .$$

For this series, $\mu = -1$ but

$$nE(n) = -\frac{n}{\ln n} + \frac{1}{2\ln n} + \frac{1}{6n\ln n} + \cdots$$

which does not have a lower bound. Gauss's test shows that this series converges at $x = -1$ where the series alternates. The test is inconclusive about what happens at $x = 1$.