

Sixteenth Annual Konhauser Problemfest

Macalester College

Saturday, Feb. 23, 2008

Problems by Bill Sands (University of Calgary)

Calculators of any sort are allowed. Justifications are expected, not just the statement of an answer. Partial credit will be given for progress toward a solution or the answer to part of a question.

1. Leave a Tip I was at a restaurant the other day. The bill came and I wanted to give the waiter a whole number of dollars, with the difference between what I give him and the bill being the tip. I always like to tip between 15% and 20% of the bill. But if I gave him a certain number of dollars, the tip would have been **less** than 15% of the bill, and if instead I gave him one dollar more, the tip would have been **more** than 20% of the bill. What was the largest possible amount of the bill?

2. Dedicated to Ramanujan The number 1729 has the property that it is divisible by the sum of its digits. In fact, 1728 has this property as well. Find three consecutive numbers larger than 1729 such that each is divisible by the sum of its decimal digits.

3. Intersection Chance Let p and q be two positive real numbers less than 1 and consider a square of side 1 whose sides are horizontal and vertical. A horizontal line of length p is drawn at random inside the square (the line cannot stick outside the square). Independently, a vertical line of length q is drawn at random inside the square. What is the probability that the two lines intersect?

4. Tip Your Glass A cylindrical glass about one-third filled with water is sitting on a level table. It is then tipped slightly so that its circular base touches the table at one point. Has the distance of the surface of the water from the table increased, decreased, or stayed the same? (Note: Assume the glass forming the bottom of the glass has zero thickness.)

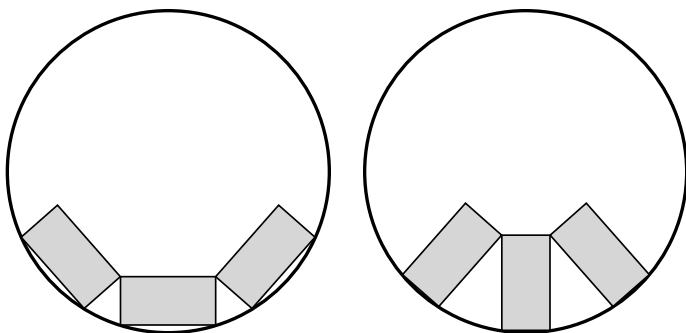
5. Old Idaho Usual Here Let w be any n -letter word ($n \geq 1$) containing at most 10 different letters, like *KONHAUSER*, *PROBLEMFEST*, *OLDIDAHO*, or *ZZZZZZZZZZZZZZZZ*. Prove that you can replace the letters of w by decimal digits (different letters replaced by different digits) so that the resulting n -digit number is a multiple of 3.

6. Hilbert College In Hilbert College there are an infinite number of lockers, numbered by the natural numbers: $1, 2, 3, \dots$. Each locker is occupied by exactly one student. College President Cantor decides to rearrange the students so that the lockers are all still occupied by the same set of students, one to a locker, but in some other order. (Some students might not change lockers.) It turns out that the locker numbers of infinitely many students end up higher than before. Show that there are also infinitely many students whose locker numbers are lower than before.

7. Integers as Dot Products Let $P(k)$ be the assertion: Every integer is a dot product of two arithmetic progressions of integers, each of length k . So, for example, $P(3)$ is true if and only if every integer n has the form $a_1 b_1 + a_2 b_2 + a_3 b_3$, where $\{a_i\}$ and $\{b_i\}$ are arithmetic progressions. Determine the truth or falsity of each of $P(3)$, $P(4)$, $P(5)$, and $P(6)$.

Note: An *arithmetic progression of integers* refers to any sequence of the form $a, a + b, a + 2b, a + 3b, \dots$, where a and b are arbitrary integers..

8. Puzzle of the Round Table A cafeteria at a large university has round tables (of every possible radius) and rectangular trays (nonsquare and all the same size). Diners place their trays of food on the table so that two adjacent corners of each tray are on the edge of the table, and with either the short or long sides of the trays pointing toward the center of the table. Moreover, at the same table everybody aligns their trays the same way. Suppose n mathematics students ($n \geq 2$) come in to eat together. How should they align their trays so that the table needed is as small as possible?



9. Find a Transversal Let X be the set of three numbers $\left\{0, \frac{4}{11}, 1\right\}$. Find a set of numbers that intersects every translation of X by a real number in exactly one number. Note: A translation of X by t is the set $\left\{t, \frac{4}{11} + t, 1 + t\right\}$.

10. Sort the Cards You are given three cards. Each card has three nonnegative real numbers written on it from top to bottom, with sum 1. Show that you can put the three cards in some order so that the sum of the first number on the first card, the second number on the second card, and the third number on the third card lies in the closed interval $\left[\frac{1}{2}, \frac{3}{2}\right]$.

