

Hints, Comments, and Solutions for Exercises in
Proofs and Confirmations
 July 13, 1999

This is a work in progress. Comments and corrections are welcome and will be added with grateful attribution.

1.1.5 This turned out to be much more difficult than I anticipated. It is easy to see that any subset that contains all permutation matrices cannot contain any ASMs with any -1 's (otherwise, we can permute the columns so that we get a -1 in the first column). I believe that given any ASM with at least one -1 , some power of it must fail to be an ASM. Heuristically, taking powers increases the number of -1 s and/or moves the -1 s closer to the boundary, but I would very much appreciate a simple proof that any ASM with a -1 in it has a power that fails to be an ASM.

1.1.12 The conjecture is that $A_{n,k}$ can be odd if and only if $n = (4^t - 1)/3$, $(4^t + 2)/3$, $(2 \cdot 4^t + 1)/3$ or $(2 \cdot 4^t + 4)/3$ for some integer $t \geq 1$. If $n = (4^t - 1)/3$, then $A_{n,k}$ is odd if and only if $k = (n - 1)/2$, $(n + 1)/2$, or $(n + 3)/2$. If $n = (2 \cdot 4^t + 1)/3$, then $A_{n,k}$ is odd if and only if $k = (n + 1)/2$. Let $A(t)$ (respectively, $B(t)$) be the set of values of k for which $A_{n,k}$ is odd when $n = (4^t + 2)/3$ (respectively, $n = 2(4^t + 2)/3$). Then $A(t) = \{k | k \in A(t - 1) \text{ or } n + 1 - k \in A(t - 1)\}$ (respectively, $B(t) = \{k | k \in B(t - 1) \text{ or } n + 1 - k \in B(t - 1)\}$).

n	k
1	1
2	1, 2
3	2
4	1, 4
5	2, 3, 4
6	1, 2, 5, 6
11	6
12	1, 4, 9, 12
21	10, 11, 12
22	1, 2, 5, 6, 17, 18, 21, 22
43	22
44	1, 4, 9, 12, 33, 36, 41, 44
85	42, 43, 44
86	1, 2, 5, 6, 17, 18, 21, 22, 65, 66, 69, 70, 81, 82, 85, 86
171	86
172	1, 4, 9, 12, 33, 36, 41, 44, 129, 132, 137, 140, 161, 164, 169, 172

2.2.11 This is the Jacobi triple product identity with q replaced by q^m and then $x = -b^{-a}$.

2.2.10 This is the Jacobi triple product identity with q replaced by q^{-1} , and then

2.2.9 If there is a 0 in the second partition, we remove it and then interchange the partitions. If there is no 0 in the second partition, we append a 0 to the first partition and then interchange them.

2.2.8 Add one to each of the parts in the second partition, subtract one from each part in the first partition, and then interchange the two partitions.

2.2.7 Hint: In the partition with no multiples of d , write the number of times each integer appears as a sum of powers of d multiplied by coefficients that are less than or equal to $d - 1$.

$$2.2.3 \quad 17 + 17 + 15 + 1$$

2.2.2 $\delta(\lambda)$ is the size of the largest square that will fit inside the partition; the respective values are 3, 3, and k . $\sigma(\lambda)$ is the number of consecutive parts, starting with the largest, that differ by exactly 1; the respective values are 2, 2, and k .

2.1.18 This is Fabian Franklin's proof which he discovered while in Sylvester's class on partition theory. Given a partition $\lambda = (\lambda_1 > \lambda_2 > \dots \geq \lambda_p > 0)$ into distinct parts, let $\sigma(\lambda)$ be the largest integer such that $\lambda^{i-1} - \lambda_i = 1$ for $i \leq \sigma$. If $\lambda_1 - \lambda_2 > 1$, then $\sigma = 1$. If $\lambda_p \leq \sigma$, then we delete the part λ_p and add one to each of the r largest parts. If $\lambda_p > \sigma$, then we subtract one from each of the *sigma* largest parts and add a new part of size σ . This is an involution on the set of partitions of n into distinct parts that changes the parity of the number of parts. The only time it cannot be applied is when $\sigma = r = \lambda_p$ or $\sigma = r = \lambda_p - 1$. The former is a partition of $r^2 + r - 1$ /2 and the latter is a partition of $r^2 + r + r - 1$ /2.

$$2.1.17 \quad \prod_{i=1}^{\infty} \frac{1 - q^{d^i}}{1 - q^{d^i}} = \prod_{i=1}^{\infty} \left(1 + q^i + q^{2i} + \dots + q^{(d-1)i} \right).$$

If $j \neq k$, then we can divide both sides by $j - k$ to get that $3(j + k) = \pm 1$. This cannot be since three divides the left side but not the right. If the signs are different, we can write the equality as $3(j_2^2) = \pm(j + k)$. Since j and k are positive, $j + k$ is not zero and so we can divide by it: $3(j - k) = \pm 1$. again, this cannot be.

$$3(j_2^2 - k_2^2) = \pm(j - k).$$

2.1.10 First assume both signs are the same, then we can write the equality as

$$\begin{aligned} & \cdot (i)^\sigma = \\ & (i)^\sigma + \left(\sum_{l=1}^{i-1} (i)^\sigma - 2^l \right) = \\ & \sum_{d=0}^{i-1} \sum_{k|d} 2^k (i-1)^k = \sigma_{\text{alt}}(i) \end{aligned}$$

On the other hand
To prove this directly, write $j = 2^v \cdot i$ where i is odd, then $\sigma^{\text{odd}}(j) = \sigma(i)$.
The proof is then by induction, using the fact that $d \circ d(n) = p \circ d(n)$.

$$\begin{aligned} \sigma_{\text{alt}}(n) &= \sum_{u=1}^{n-1} n p d u - \sum_{u=1}^{n-1} \sigma_{\text{alt}}(j) d p u \\ \sigma^{\text{odd}}(n) &= \sum_{u=1}^{n-1} n p d u - \sum_{u=1}^{n-1} \sigma^{\text{odd}}(j) d p u \end{aligned}$$

2.3.6 Exercises 2.3.4 and 2.3.5 imply that

$$\begin{aligned} & \cdot \left(\sum_{p=1}^{\infty} b(p)^{\text{odd}} \right) \left(\sum_{u=1}^{\infty} d(u) \circ d \right) = \\ & \sum_{m=1}^{\infty} d(u) \circ d \prod_{l=1}^m \frac{1 - 2^l}{1 - 2^{2^l}} = \\ & \sum_{m=1}^{\infty} \prod_{l=1}^m \frac{1 - 2^l}{1 - 2^{2^l}} = \sum_{m=1}^{\infty} \prod_{l=1}^m \frac{1 - 2^l}{1 - 2^{2^l}} \end{aligned}$$

2.3.4 Taking the derivative of the generating function, we get

2.3.2 No.

$$-2q^{-2} \prod_{k=1}^{\infty} (1 + q^k) - \prod_{k=1}^{\infty} (1 - 2q^{-2k}) = -2q^{-2} \prod_{k=1}^{\infty} (1 + q^k) - \prod_{k=1}^{\infty} (1 - 2q^{-2k})$$

2.2.15 Equation (2.13) simplifies to

second equality: $b_{-1} = -b_0$.

2.2.13 Using the first equality and then the second: $b_1 = b_{-2} = -b_1$. From the

$$\begin{aligned} g(x)_{-1} &= g(x) \frac{1-x}{1-x} = g(x) \cdot \\ g(x/q) &= g(x) \frac{(1-x)(1-x)}{(1-x)(1-x)} = g(x) \cdot \end{aligned}$$

2.2.12

3.1.1 Conjugate the Ferrers graph.

$$\det \begin{pmatrix} x_i + j & & \\ & \ddots & \\ & & x_i + j \end{pmatrix} = (-1)^{\binom{n}{2}} \prod_{i=1}^n \frac{x_i^{n+1-i}}{x_i + 1} \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

The determinant on the right is an alternating polynomial in the x_i of total degree $n(n-1)/2$ and so it is equal to the Vandermonde product times a constant which must be $(-1)^{\binom{n}{2}}$.

$$\det \begin{pmatrix} x_i + j & & \\ & \ddots & \\ & & x_i + j \end{pmatrix} = \prod_{i=1}^n \frac{x_i^{n+1-i}}{x_i + 1} \det \left((x_i + 2)(x_i + 3) \cdots (x_i + j) \right).$$

2.4.7

2.3.18 $MT_n(\{1, i\}) = MT_{n-1}(i)$.

2.3.15 It is the sum over all rows, except the bottom row, of the number of integers in that row that do not appear in the row below.

$$2.3.14 \text{ (a)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

2.3.11 Given the power series $1 + \sum_{r=1}^{\infty} b^r q^r$, let $a_1 = b_1$. The coefficient of q^0 and q^1 agree in the power series with those in the expansion of $(1 - q)^{-b_1}$. Assume that we have found unique integers a_1, \dots, a_{j-1} such that the coefficients of q^0, q^1, \dots, q^{j-1} in the expansion of $\prod_{i=1}^{j-1} (1 - q^{-a_i})$ agree with those in the power series, and let c_j be the coefficient of q^j in the expansion of this product. the coefficient c_j is an integer. Let $a_j = b_j - c_j$. Check that the coefficient of q^j in $\prod_{i=1}^j (1 - q^{-a_i})$ is equal to b_j .

where $b_j = 1$ if 6 does not divide j and $b_j = -5$ if 6 divides j .

$$a^k(n) = \frac{n-k}{k} \sum_{j=1}^{n-k} b_j a^k(n-j)$$

We have that $a^k(n) = 0$ for $n < k$, $a^k(k) = 1$, and for $n > k$:

$$2.3.7 \quad \sum_{n=1}^{\infty} n a^k(n) x^n = k \left(\sum_{n=1}^{\infty} a^k(n) x^n \right) \left(\frac{1-x}{1-\frac{x}{6x^6}} \right).$$

$$\prod_{i=1}^{\ell} (1 - b^{-i+u}) = \prod_{i=1}^{\ell} (1 - b^{-i})$$

3.1.20

$$x^i b^i \begin{bmatrix} i \\ n+i-1 \end{bmatrix} = \frac{1}{1 - b^{-i}}$$

3.1.19 The coefficient of x^i is the generating function for partitions into exactly i parts, all of which are less than or equal to n . If we subtract 1 from each of these parts, we are left with the generating function for partitions into at most i parts, each of which is $\leq n-1$, and this generating function is $\begin{bmatrix} i \\ n+i-1 \end{bmatrix}$. It follows that

$$\frac{(b-1)}{(1-b^{-2})} + \frac{(b-1)(b^{-2})}{(1-b^{-2})(1-b^{-1})} = \frac{(b-1)(b^{-2})}{(1-b^{-2})(1-b^{-1})}$$

$$= \begin{bmatrix} 2 \\ n+2 \end{bmatrix} = \begin{bmatrix} 2 \\ n \end{bmatrix} + \begin{bmatrix} 0 \\ n \end{bmatrix} + \begin{bmatrix} 1 \\ 2n-1 \end{bmatrix}$$

3.1.13 For $j=2$, this identity is

3.1.10 Note that the conditions should read "positive for $k \equiv \pm 1 \pmod{8}$ and negative for $k \equiv \pm 3 \pmod{8}$." The sign of the product is positive if there are an even number of integers j in the interval $(k/4, (k+1)/2]$ and negative if there are an odd number of integers in this interval.

$$3.1.6 \alpha^{(k+1)/2} = \alpha^{(k-1)/2} = \alpha^{1+3+\dots+k-2}$$

residues modulo k .

3.1.5 $\alpha^{j^2+j} = \alpha^{(k+1)j}$ and the sum in $G(\alpha)$ can be taken over any set of

3.1.4 Note that $\alpha^k = 1$.

$$f(g, m) = 1 - \begin{bmatrix} 1 \\ m-1 \end{bmatrix} z^{-m} + \begin{bmatrix} 2 \\ m-1 \end{bmatrix} z^{-2m} + \dots + \begin{bmatrix} m-2 \\ m-2 \end{bmatrix} z^{-(m-2)m}$$

$$= \sum_{m=1}^{\ell} (-1)^{j-1} (1 - b^{-m}) \begin{bmatrix} m-j \\ m-1 \end{bmatrix}$$

$$= \sum_{m=1}^{\ell} (-1)^{j-1} (1 - b^{-m}) \begin{bmatrix} m-j \\ m-2 \end{bmatrix} = \prod_{i=1}^{\ell} (1 - b^{-i+u})$$

3.1.3 Using the recursive formula, we have that

Once we have placed the $j - 1$ st k into the sequence in such a way that it increases the major index by λ_{j-1} , we placed the next k as follows. If $\lambda_j - j \geq 0$, then we find the first non-descent to the left of the last placed k where inserting a k increases the total major index by exactly λ_j . Note that such a position always exists because placing the k in the position to the left of the entry immediately to the left of the last k will increase the major index by exactly λ_{j-1} if it has not been placed inside a pre-existing descent. If this would put it inside a pre-existing descent, then putting this new k in the first position to left of that position that is not a pre-existing descent will increase the major index by exactly λ_{j-1} . We

Note that wherever we put this first k , we have increased the major index first position to the left where it does not fall inside a pre-existing descent. Then insert k after the entry in position $\lambda_1 - 1$. Otherwise, insert k in the entry in σ in position $\lambda_1 - 1$ is less than or equal to the next entry, We begin with the largest part in $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{m_k} \geq 0)$. If

$$\begin{aligned} \sum_{\text{maj}(\tau)} b^{\tau} &= \sum_{\sigma} b^{\sigma} \sum_{|\lambda|} \\ &= \begin{bmatrix} m_1 + \dots + m_k \\ m_1, \dots, m_k \end{bmatrix} \\ &= \begin{bmatrix} m_1 + \dots + m_k \\ m_1, \dots, m_k \end{bmatrix} \begin{bmatrix} M \\ m_k \end{bmatrix} \end{aligned}$$

3.2.4 We proceed by induction on the number of variables. The previous exercise established this result for $k = 2$. Assume that it is correct up $k - 1$, $k \geq 3$. Let σ be a sequence with m_i 's, $1 \leq i < k$, and let λ be a partition into at most m_k parts less than or equal to $M = m_1 + \dots + m_{k-1}$. We must establish a one-to-one correspondence between the set of pairs (σ, λ) and the set of sequences τ , with m_i 's, $1 \leq i \leq k$, such that $\text{maj}(\sigma) + |\lambda| = \text{maj}(\tau)$. It then follows that

$$3.2.1 \text{ maj}(110010100) = 2 + 5 + 7 + 9 = 23. \mathcal{I}(01101011001) = 2 + 3 + 5 + 5 = 15. \mathcal{I}(110010100) = 2 + 2 + 3 + 4 + 5 + 5 = 21.$$

$$3.1.21 \quad \lim_{n \rightarrow \infty} \begin{bmatrix} n \\ i \end{bmatrix} = \lim_{n \rightarrow \infty} \frac{(b^{-n-i+1}; b)_i}{(b^{-n-i+1}; b)_i} = \frac{1}{1}.$$

Note that if $i < n$, then one of the factors in $(b^{-n}; b)_i$ is $1 - 1 = 0$.

$$\begin{bmatrix} n \\ i \end{bmatrix} b^{i(-i-1)/2} (x^{-i}) = \frac{(b^{-n-i+1}; b)_i}{(b^{-n-i+1}; b)_i} b^{i(-i-1)/2} (x^{-i}) = \frac{(b^{-n-i+1}; b)_i}{(b^{-n-i+1}; b)_i} (x^{-i}) b^{i(-i-1)/2} (x^{-i})$$

It follows that

$$\prod_{i=1}^z \frac{1 - d^{i+j+k-2}}{1 - d^{i+j+k-1}} = \prod_{\substack{1 \leq i \leq z \\ 1 \leq j \leq r}} \frac{1 - d^{i+j+k-1}}{1 - d^{i+j+k-2}} \leftarrow \prod_{i=1}^z \frac{1 - d^{i+j+k-1}}{1 - d^{i+j+k-2}}.$$

3.3.1

3.2.15 For each product of entries that is -1 , there are two uniquely defined products that are $+1$: If there is a $+1$ to the northeast of a -1 , then there is a $+1$ directly south of that -1 and another $+1$ directly west; if there is a -1 northeast of a $+1$, then there is another $+1$ directly north of that -1 and another $+1$ directly east.

3.2.14 $(1 + q)^{n(n-1)/2}$ (see exercise 3.5.9).

3.2.12 See page 193 for what is known about these polynomials in k , and exercises 6.1.3 and 6.1.4 for some conjectures.

3.2.11 Nothing is known about these polynomials. They appear to be monotonic.

3.2.9 If $k \in r^{z+1} - r^z$, then there is a 1 in row $i + 1$, column k . The set $[[k - r^{z+1}]]$ is the subset of $\{1, \dots, k - 1\}$ for which these columns have more 1 s than -1 s in the rows below $i + 1$. If $k \in r^z - r^{z+1}$, then there is a -1 in row $i + 1$, column k .

3.2.7 Choose any two of the n positions in the sequence. There are $\binom{n}{2}$ such choices. In exactly half of the permutations, $n!/2$ permutations, these two positions contribute 1 to the inversion number.

3.2.6 The inversion number of a permutation of n letters is equal to the number of positions to the right of n (which equals the number of transpositions required to move n to the last position on the right) plus the inversion number of the word in $n - 1$ letters obtained by removing that n .

3.2.5 One proof is to assign to each permutation σ the companion, σ' , obtained by interchanging the images of 1 and 2 : $\mathcal{I}(\sigma') = \mathcal{I}(\sigma) \pm 1$. Another is to set $q = -1$ in equation (3.20).

by which the major index decreases.
 left to right, removing all remaining k s and keeping track of the amount the major index decreases with each removal. Then we read the word from those k s that sit inside descents and keeping track of the amount by which To reverse this bijection, we read the word from right to left, removing

If $\lambda_j > j$, then we place the j th inside the λ_j th descent as counted from the right. This increases the major index by exactly λ_j .
 index is exactly λ_j .
 now keep moving to the left until the amount that we increase the major

$$\left[\prod_{1 \leq i < j \leq n} (x_i + x_j) \prod_{1 \leq i < j \leq n} (x_i - x_j) \right]^\chi = |x_{n-1}^\chi|$$

3.5.6

3.5.3 the proof exactly follows that of theorem 3.12 except that M^* is constructed by replacing the last $n - m$ columns of M^C by the last $n - m$ columns of the identity matrix.

$$\begin{pmatrix} 4 & -2 & -5 \\ 1 & 3 & 5 \\ 2 & -1 & 11 \end{pmatrix} \leftarrow \begin{pmatrix} 7 & 7 \\ 7 & 38 \end{pmatrix} \leftarrow (77).$$

3.5.1

3.4.7 Same proof as exercise 3.4.6, but instead of just counting the number of lattice paths, keep track of the partition that each lattice path encodes.

3.4.6 Let $j - k$ be the y -coordinate of the first point on the path that touches the line $x = r - 1$.

$$\begin{aligned} \det A &= \sum_{\sigma \in S_n} (-1)^{I(\sigma)} \prod_{n-1}^{\sigma} a_{i, \sigma(i)} \\ &= \sum_{\sigma \in S_n} (-1)^{I(\sigma)} \left(\prod_{n-1}^{\sigma} a_{i, \sigma(i)} \right) \left(\sum_{i=1}^k a_{i, \sigma(i)} \right) \\ &= \sum_{i=1}^k \sum_{\sigma \in S_n} (-1)^{I(\sigma)} \prod_{n-1}^{\sigma} a_{i, \sigma(i)} \\ &= \sum_{i=1}^k \det A_i. \end{aligned}$$

3.4.5 Let $A = (a_{ij})_{i,j=1}^n$ and $a_{nj} = \sum_{i=1}^k a_{ij}$. Then

Therefore, f is such a function. Since every permutation can be built from adjacent transpositions, such a function is unique.

$$f(\sigma) - f(\sigma') = i(i-1) \text{sgn}(\sigma) - (i+1)(i+2) \text{sgn}(\sigma') = (i-1) \text{sgn}(\sigma) - (i+1) \text{sgn}(\sigma').$$

3.3.3 Let σ' agree with σ except that $\text{sgn}(\sigma') = \sigma(i+1) \text{sgn}(\sigma)$. Then we have

There are exactly k representations of $k+1$ as a sum of two positive integers (counting order).

4.1.9 Note correction: Last term in the displayed summation should be $(-1)^{h_{a+b+1}}$. Use equation (4.4) and expand on the first row.

$$e_0 h_n - e_1 h_{n-1} + e_2 h_{n-2} - \dots + (-1)^n e_n h_0 = 0.$$

It follows from the generating functions that for $n \geq 1$:

$$\det(e_{1-i+j}) = e_1 h_{n-1} - e_2 h_{n-2} + \dots + (-1)^{n-1} e_n h_0.$$

(4.7), expand on the first row and use induction:

4.1.8 Equation (4.6) is Proposition 4.2 with λ equal to k is. To prove equation

4.1.7 Use the generating function and Theorem 3.3 on page 78.

4.1.6 Use the generating function and the result of exercise 3.1.19 on page 83.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & & & & \\ 0 & 0 & 1 & 2 & 6 & & & & \\ 0 & 1 & 2 & 5 & 12 & & & & \\ 1 & 4 & 6 & 12 & 24 & & & & \end{pmatrix}.$$

4.1.5 Invert the matrix

of the inversion number.

4.1.3 This follows from the fact that every permutation is a composition of adjacent transpositions and an adjacent transposition changes the parity

4.1.2 Using the generating functions, e_n is the coefficient of t^n in $(1+t)^m$, which is $\binom{n}{m+n-1}$, and h_n is the coefficient of t^n in $(1-t)^{-m}$, which is $\binom{n}{m+n-1}$.

follows.

4.1.1 $7 + 6 + 6 + 5 + 1 + 1 + 1 + 1 + 1 + 1$ precedes and $7 + 6 + 6 + 4 + 4 + 1 + 1$

3.5.7 If $B_{ij} = 1$ and there are no -1 s directly below it, then j appears $n-i+1$ times in the monotone triangle. If $B_{ij} = 1$ and $B_{kj} = -1$ with $B_{kj} = 0$ for $i < k < j$, then j appears $k-i$ times in the monotone triangle and then disappears as you read down successive rows. Therefore $\sum_i (n-i) B_{ij}$ is one less than the number of times j appears.

$$\begin{aligned} & \prod_{1 \leq i < j < n} (x_i + \lambda x_j) \prod_{1 \leq i < j < n} (x_i + \lambda x_n) \\ & \div \left[x_2 x_{n-1} \dots \prod_{1 \leq i < j < n} (x_i + \lambda x_j) \right] \\ & + \lambda \prod_{1 < i < j \leq n} (x_i + \lambda x_j) \cdot x_2 \dots x_n \prod_{1 \leq i < j < n} (x_i + \lambda x_j) \end{aligned}$$

4.2.5 We encode each column of the semistandard tableau as a lattice path. Lattice paths are now permitted to touch at vertices, but may not share edges. The proof proceeds exactly as in the proof of Theorem 4.3, except that we switch the tails of two paths that share a common edge.

4.2.6 Let $x_i = q^{n+1-i}$ so that an x_i in a given square of the Young diagram represents a stack of $n+1-i$ cubes above that position. Then take the limit as $n \rightarrow \infty$.

4.2.7 Let $x_i = q^{2n+1-2i}$.