Calculus Before Newton and Leibniz: Part I

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"Along with Isaac Newton (1642-1727), Leibniz is generally credited as one of the inventors or discoverers of calculus."
-- Thomas P. Dick and Charles M. Patton, *Calculus*.

History has a way of focusing credit for any invention or discovery on one or two individuals in one time and place. The truth is not as neat. Most of the time, this doesn't matter, but in calculus, it does. When we convey the impression that Newton and Leibniz created calculus out of whole cloth, we do our students a disservice. We present mathematicians as creatures of an entirely different level of mental ability. Newton and Leibniz were brilliant, but not even they were capable of inventing or discovering calculus.

We also miss out on some great stories. The body of mathematics we know as calculus developed over many centuries in many different parts of the world, not just western Europe but also ancient Greece, the Middle East, India, China, and Japan. Newton and Leibniz drew on a vastbody of knowledge about topics in both differential and integral calculus. The subject would continue to evolve and develop long after their deaths. What marks Newton and Leibniz is that they were to the first to state, understand, and effectively use the Fundamental Theorem of Calculus. Use it effectively they certainly did. No two people have moved our understanding of calculus as far or as fast. But the problems that we study in calculus -- areas and volumes, related rates, position/velocity/acceleration, infinite series, differential equations -- had been solved before Newton or Leibniz was born.

The expression of these solutions was awkward and progress was painfully slow. It took some 1,250 years to move from the integral of a quadratic to that of a fourth degree polynomial. But awareness of this struggle can be a useful reminder for us. The grand sweeping results that solve so many problems so easily (integration of a polynomial being a prime example) hide a long conceptual struggle. When we jump too fast to the magical algorithm, when we fail to acknowledge the effort that went into its creation, we risk dragging our students past that conceptual understanding.

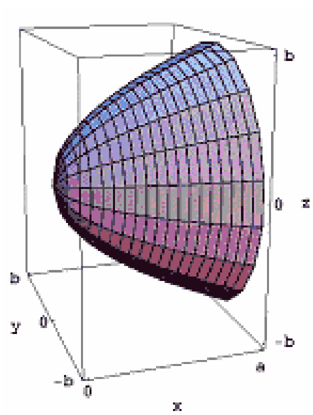
This article is the first in a series of articles that will explore the history of calculus before Newton and Leibniz: the people, problems, and places that are part of the rich story of calculus.

Abu Ali al-Hasan ibn al-Haytham (also known by the Latinized form of his name: Alhazen) was one of the great Arab mathematicians. He was born in Basra, Persia, now in southeastern Iraq. Sometime after 996, he moved to Cairo, Egypt where he became associated with the University of Al-Azhar, founded in 970. He was prolific, writing over 90 books, and is most famous for his work in astronomy and optics. His interest in mathematics ranged over algebra, geometry, and number theory. I focus on him because he is the first person I know of to have integrated a fourth-degree polynomial.

Of course, he did not express it quite that way. Around 250 B.C., Archimedes wrote *On Conoids and Spheroids*, a book that, among other things, demonstrated how to find the volume of a parabaloid, the solid of revolution that you get when you rotate a parabola around its axis (see Figure 1). In particular, if *a* and *b* are positive constants and we take the region bounded above by the graph of the parabola below by the *x*-axis, and to the right by *x* = *a* (see Figure 2), and rotate this region around the *x*-axis, we get the solid of revolution whose volume is

$$\pi\int_0^a \left(b\sqrt{x/a}\right)^2 dx = \pi\frac{b^2}{a}\int_0^a x\,dx = \frac{\pi a b^2}{2}.$$

In other words, the volume is exactly half that of the cylinder that you get if you rotate the rectangle of length *a* and height *b* around the *x*-axis.



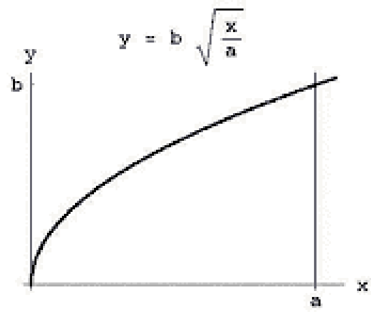
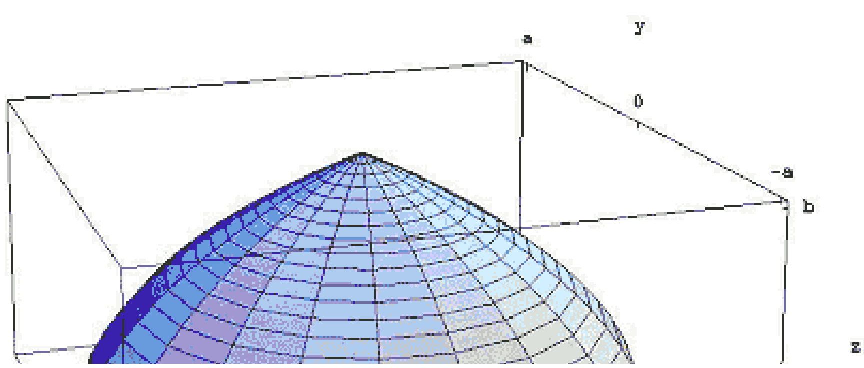


Figure 2: The region bounded by $y = b\sqrt{x/a}$, the *x*-axis, and $x = a$

Figure 1: The parabaloid.

The difficult part of this calculation, something that it took a mathematician of Archimedes' stature to realize, is that the problem of finding the volume of a parabaloid can be reduced to that of finding the area under a straight line (the integral of *x* from 0 to *a*).

In the Arab world of the tenth century, Archimedes' *On Conoids and Spheroids* was unknown, but Thabit ibn Qurra of southern Turkey and Abu Sahl al-Kuhi of northern Iran had discovered their own proofs of the volume of a parabaloid. Ibn al-Haytham read their work and asked himself the question: What if we rotate this region around the line *x* = *a* instead of the *x*-axis?



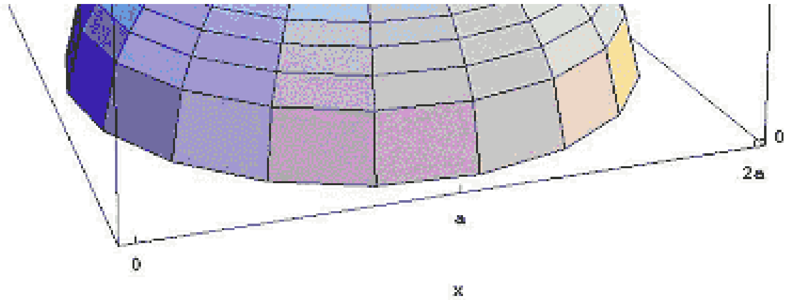


Figure 3: Rotating the region around the line *x* = *a*.

The result is the very Islamic-looking dome shown in Figure 3. Ibn al-Haytham showed that its volume is 8/15ths of the volume of the cylinder that you get when you rotate the rectangle of length *a* and height *b* around *x* = *a*. In the notation of modern calculus, the calculation of this volume becomes

$$\pi\int_0^b\left(a-\frac{ay^2}{b^2}\right)^2dy=\frac{\pi a^2}{b^4}\int_0^b\left(b^2-y^2\right)^2dy=\frac{\pi a^2}{b^4}\left[b^4y-\frac{2}{3}b^2y^3+\frac{1}{5}y^5\right]_0^b=\frac{8}{15}\pi a^2b$$

But ibn al-Haytham lived almost 700 years before the formulas for integrals would be known. He found the volume by stacking disks. If we slice the dome into *nv* disks, each of thickness *b/n*, then the *i*th disk from the bottom has radius $a - ai^2/n^2$ and volume $(b/n)\ \pi\ (a - ai^2/n^2)^2$ (see Figure 4). The total volume of this stack of disks is

$$\sum_{i=1}^{n}\frac{b}{n}\pi\left(a-a\frac{i^2}{n^2}\right)^2=\frac{\pi a^2b}{n^5}\sum_{i=1}^{n}\left(n^2-i^2\right)^2$$

.

All that's left is to find a formula -- in terms of *n* -- for the summation. We then see what happens as *n* approaches infinity. We expand the summand and pull out the constant powers of *n*:

$$\sum_{i=1}^{n}\left(n^2-i^2\right)^2=\sum_{i=1}^{n}\left(n^4-2n^2i^2+i^4\right)=n^5-2n^2\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i^4$$

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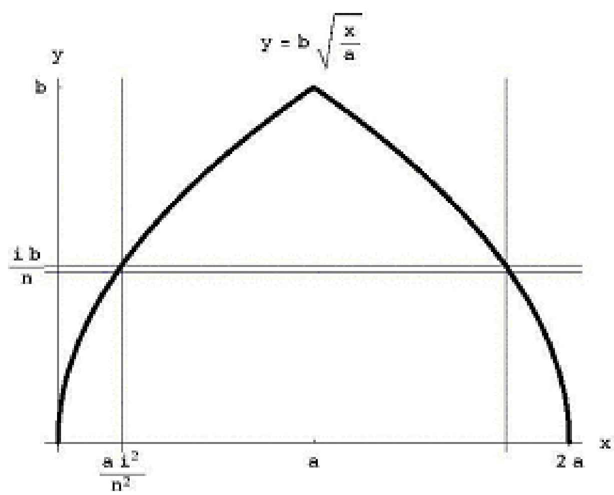

Figure 4: The horizontal slice across the dome.

For ibn al-Haytham as for mathematicians around the Mediterranean, through the Middle East, South Asia, and East Asia, the problem of calculating areas and volumes came down to the problem of finding sums of powers of consecutive integers. Ibn al-Haytham was one of many mathematicians in many different places who succeeded in solving this problem. He showed that:

$$\sum_{i=1}^{n}\left(n^2-i^2\right)^2=\frac{8}{15}n^5-\frac{1}{2}n^4-\frac{1}{30}n,$$

The volume of the dome is:

$$\lim_{n\to\infty}\pi a^2 b\left(\frac{8}{15}-\frac{1}{2n}-\frac{1}{30n^4}\right)=\frac{8}{15}\pi a^2 b.$$

How did he find this summation? That's another story; one that I'll tell next time. It spans over two thousand years and three continents.

Sources:

1. Dick, Thomas P. and Charles M. Patton. *Calculus*. PWS Publishing, 1995.
2. Katz, Victor J. "Ideas of Calculus in Islam and India"; *Mathematics Magazine*, 1995. (68-3). 163-174.
3. Katz, Victor J. *A History of Mathematics: An Introduction*. 2nd edition. HarperCollins, 1998.