The Benefits Of Bijections

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MAA

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Joint Math Meetings
Boston, MA
January 5, 2012

Ira Gessel's Bijective Proof of the Vandermonde Determinant Formula (*J. Graph Theory*, 1979)

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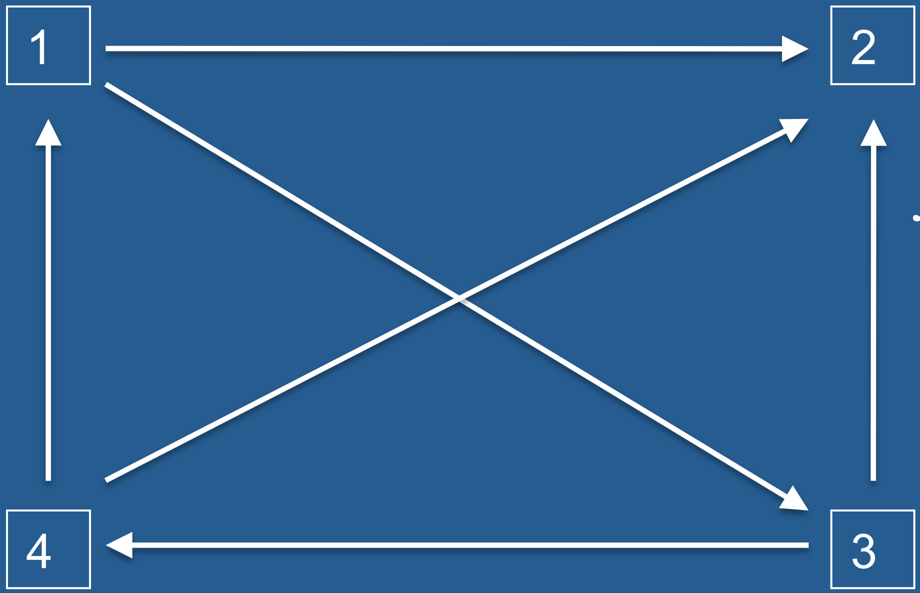

$$x_1^2\left(-x_3\right)x_3\left(-x_4\right)^2$$

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$$\sum_{T\in\mathcal{T}_n}(-1)^{u(T)}\prod_{i=1}^{n}x_i^{w(i)} = \sum_{\sigma\in S_n}(-1)^{\operatorname{sgn}(\sigma)}\prod_{i=1}^{n}x_i^{n-\sigma(i)}$$

$u(T)$ = # of upsets in the tournament

$w(i)$ = # of wins by player i

A tournament is *transitive* if it defines a permutation.

Vandermonde determinant formula is equivalent to a weight-preserving, sign-reversing involution on non-transitive tournaments.

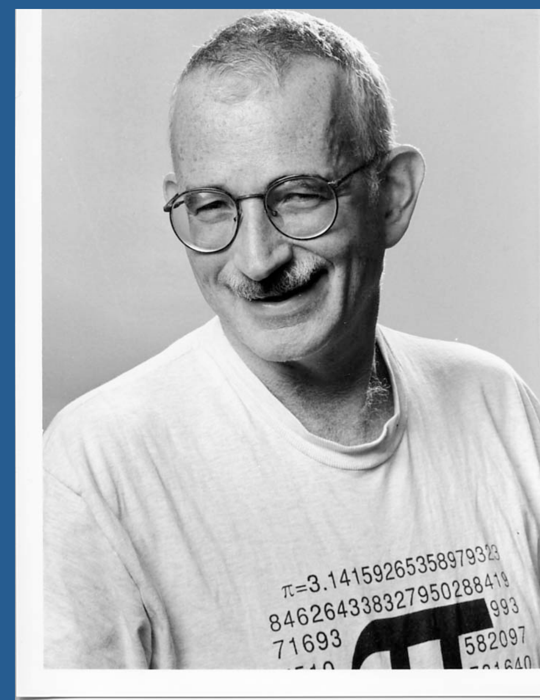
Tournament is transitive if and only if the vertices have distinct out-degrees: 0, 1, …, $n-1$.

Given non-transitive tournament, $T_{1,}$ find the first pair of vertices with the same out-degree, say (i, j).

Define T_2 by keeping same directed arrows for vertices other than i or j. In T_2, i—>k if and only if j—>k in T_1. In T_2, j—>k if and only if i—>k in T_1. Reverse the arrow between i and j.

The number of wins by each player stays the same. For each $k \neq i$ or j, the number of upsets involving k stays the same or possibly changes by 2 (if $i <$ k $< j$ and i —> k —> j or i <— k <— j). The number of upsets between i and j changes by 1.

Doron Zeilberger's Bijective Proof of the Dyson Conjecture (*Discrete Math*, 1982)



Conjectured 1962, proven by Gunson (1962), Wilson (1962), Good (1970).

$$\mathrm{CT} \prod_{1 \le i \ne j \le n} \left(1 - \frac{x_j}{x_i}\right)^{a_i} = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1!\, a_2! \cdots a_n!}$$

$$\left[\left(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\right)^{n-1}\right] \prod_{1 \le i < j \le n} \left(x_i - x_j\right)^{a_i + a_j} = (-1)^K \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1!\, a_2! \cdots a_n!}$$

$$K = a_2 + 2a_3 + \cdots + (n-1)a_n$$

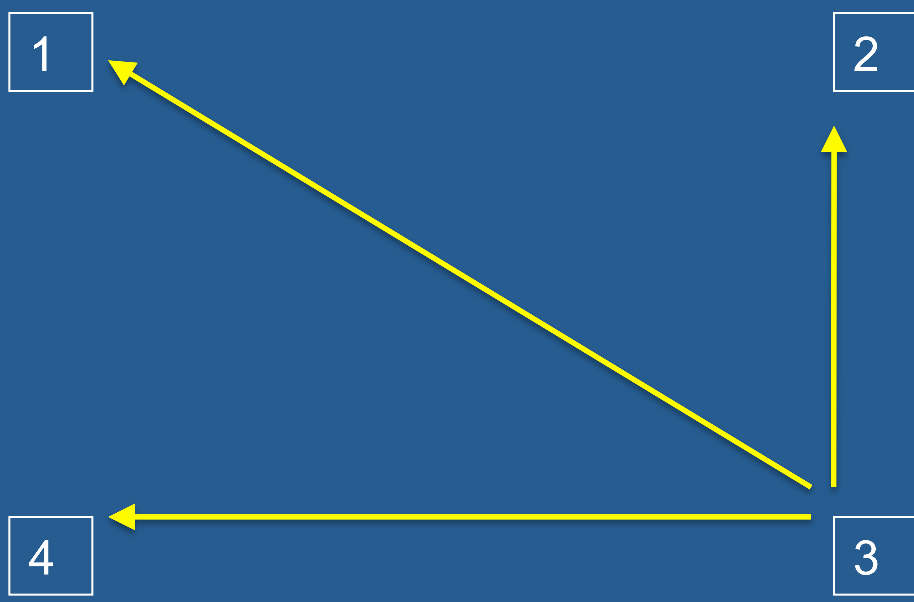
$$\left[\left(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}\right)^{n-1}\right]\prod_{1\le i<j\le n}\left(x_i-x_j\right)^{a_i+a_j}=(-1)^K\frac{(a_1+a_2+\cdots+a_n)!}{a_1!\,a_2!\cdots a_n!}$$

LHS counts number of multi-tournament, i, j play $a_i + a_j$ games. Player i wins $(n–1)\,a_i$ games.

$\dfrac{(a_1+a_2+\cdots+a_n)!}{a_1!\,a_2!\cdots a_n!}$ Counts the number of words with a_1 1s, a_2 2s, … , a_n n's.

Given such a word, say 314223114, the successive outcomes of the games between i and j is the subword in i and j. The wins between players 1 and 2 are 12211. Between 2 and 3 they are 3223. If the results of the multi-tournament can be encoded as such a word, then the number of upsets is K.

314223114



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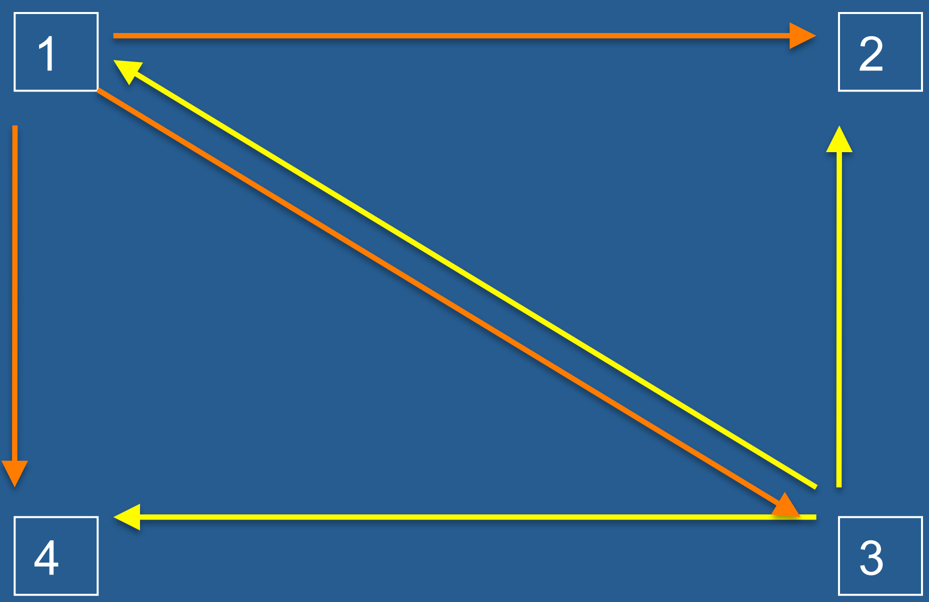

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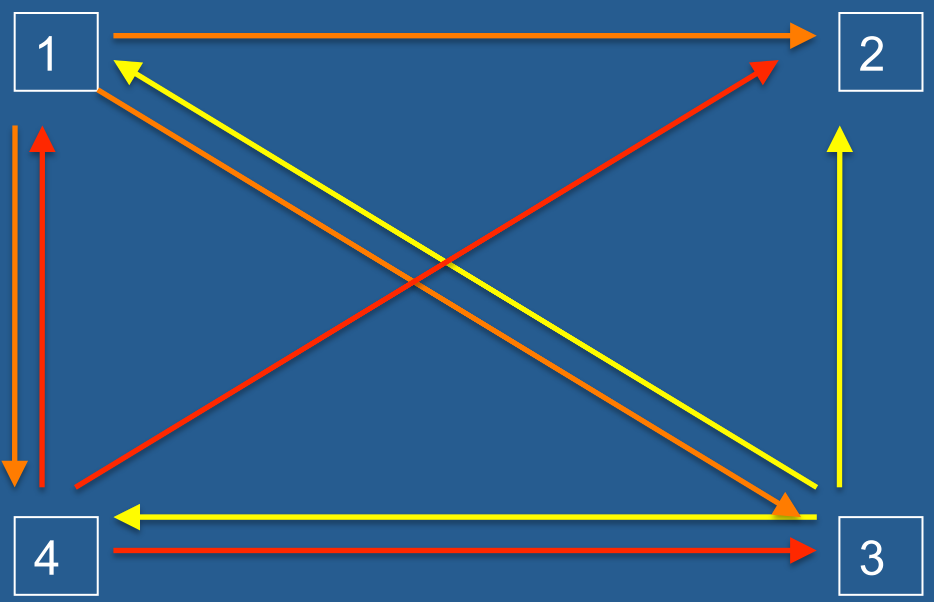

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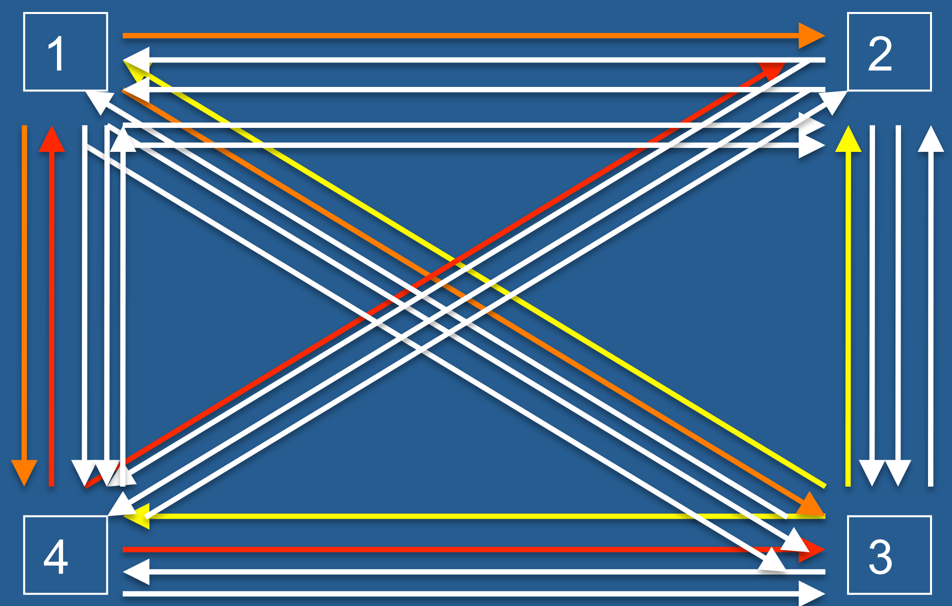
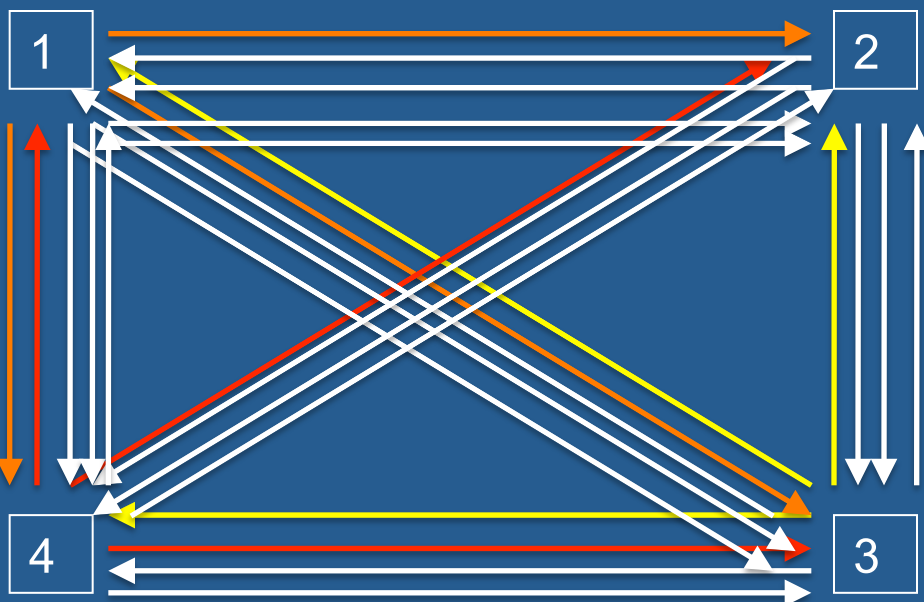
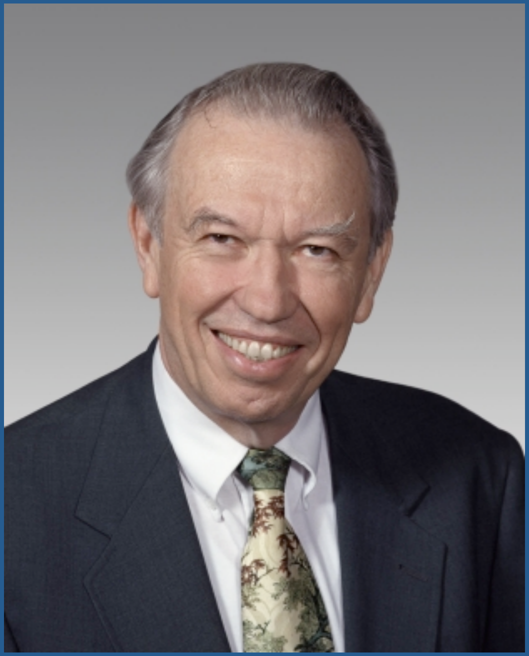


Consider tournament defined by first directed edge between each pair of vertices. If transitive, record overall winner, remove the directed edges out of that vertex, and repeat.



If it is not transitive, use the previous bijection to change it to another non-transitive tournament. Put back in the edges that have been removed.

Z and B's Bijective Proof of Andrews' q-Dyson Conjecture (*Discrete Math*, 1985)



Conjectured 1975

$$\left[\left(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}\right)^{n-1}\right]\prod_{1\le i<j\le n}\prod_{k=0}^{a_i+a_j-1}\left(x_i-x_jq^k\right)$$

$$=\left(-1\right)^K q^L\frac{\prod_{k=1}^{a_1+a_2+\cdots+a_n}\left(1-q^k\right)}{\prod_{k=1}^{a_1}\left(1-q^k\right)\prod_{k=1}^{a_2}\left(1-q^k\right)\cdots\prod_{k=1}^{a_n}\left(1-q^k\right)}$$

$$K=a_2+2a_3+\cdots+\left(n-1\right)a_n,$$

$$L=\binom{a_2}{2}+2\binom{a_3}{2}+\cdots+\left(n-1\right)\binom{a_n}{2}$$

$$\left[\left(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}\right)^{n-1}\right]\prod_{1\le i<j\le n}\prod_{k=0}^{a_i+a_j-1}\left(x_i - x_jq^k\right)$$

The t'th game between players i and j carries an additional weight of q^{t-1} if player j beats player i.

The sign-reversing bijection between multi-tournaments that do not correspond to a word gets considerably more complicated, but can be described.

The sum of all words with a_i i's weighted in this way can be shown to equal the other side of the equality:

$$(-1)^K q^L \frac{\displaystyle\prod_{k=1}^{a_1+a_2+\cdots+a_n} \left(1-q^k\right)}{\displaystyle\prod_{k=1}^{a_1}\left(1-q^k\right)\prod_{k=1}^{a_2}\left(1-q^k\right)\cdots\prod_{k=1}^{a_n}\left(1-q^k\right)}$$

After 21 years, Ira Gessel and Guoce Xin published a non-boijective proof, *Proc. Amer. Math. Soc.*, 2006

The proof is based on the observation that each side of

$$\left[\left(x_1^b x_2^{a_2} \cdots x_n^{a_n}\right)^{n-1}\right] \prod_{j=2}^{n} \prod_{k=0}^{b+a_j-1} \left(x_1 - x_j q^k\right) \prod_{2 \le i < j \le n} \prod_{k=0}^{a_i+a_j-1} \left(x_i - x_j q^k\right)$$

$$= (-1)^K q^L \frac{\prod_{k=b+1}^{b+a_2+\cdots+a_n} \left(1-q^k\right)}{\prod_{k=1}^{a_2}\left(1-q^k\right) \cdots \prod_{k=1}^{a_n}\left(1-q^k\right)}$$

is a polynomial in q^b of degree at most $a = a_2 + a_3 + \cdots + a_n$. They find a common zeros $(q^{-1}, q^{-2}, \ldots, q^{-a})$. By induction, the polynomials agree at $q^b = 1$.

$$\prod_{1\le i<j\le n}\left(x_i - x_j\right) = \det\left(x_j^{n-i}\right)_{i,j=1}^{n}$$

Each side is a polynomial in x_1 of degree n–1.

Each polynomial has roots at $x_2, x_{3, \ldots,} x_n$.

By induction, the polynomials agree at $x_1 = 0$.

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