Reflections on the
Fundamental Theorem
Of Integral Calculus

David Bressoud
St. Paul, MN

MAA Wisconsin Section
Ripon College, Ripon, WI
April 25, 2015

PowerPoint available at
www.macalester.edu/~bressoud/talks
First appearance of the *Fundamentalsatz der Integralrechnung* (*Fundamental Theorem of Integral Calculus*) in its modern form was in an appendix to a paper on Fourier series by Paul du Bois-Reymond in 1876.
1. Why the Fundamental Theorem of Integral Calculus is problematic for most students.

2. A brief history of the FTIC (and why “integral” belongs in the name).

3. Some suggestions for how to prepare students to understand it.
What is the Fundamental Theorem of Calculus?

Why is it *fundamental*?

Possible answer: Integration and differentiation are inverse processes of each other.

Problem with this answer: For most students, the working definition of integration is the inverse process of differentiation.
For any function $f$ that is continuous on the interval $[a,b]$,

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x),$$

Differentiation undoes integration

and, if $F'(x) = f(x)$ for all $x$ in $[a,b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Definite integral provides an antiderivative

Integration undoes differentiation

An antiderivative provides a means of evaluating a definite integral.
The key to understanding this theorem is to know and appreciate that the definite integral is a limit of Riemann sums:

\[ \int_{a}^{b} f(x) \, dx \text{ is defined to be a limit over all partitions of } [a,b], \]

\[ P = \{a = x_0 < x_1 < \cdots < x_n = b\}, \]

\[ \int_{a}^{b} f(x) \, dx = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}), \]

where \( |P| = \max_{i} (x_i - x_{i-1}) \) and \( x_i^* \in [x_{i-1}, x_i] \).
Riemann’s habilitation of 1854:

Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe

\[
\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

Purpose of Riemann integral:

To investigate how discontinuous a function can be and still be integrable. Can be discontinuous on a dense set of points.
Riemann’s function: \( f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2} \)

\[ \{x\} = \begin{cases} 
  x - (\text{nearest integer}), & \text{when this is } < \frac{1}{2}, \\
  0, & \text{when distance to nearest integer is } \frac{1}{2}.
\end{cases} \]

At \( x = \frac{a}{2b} \), \( \gcd(a, 2b) = 1 \), the value of the function jumps by \( \frac{\pi^2}{8b^2} \).

Darboux observed: \( \frac{d}{dx} \int_{a}^{x} f(x) \, dx \neq f(x) \).
A brief history of integration as accumulation and the Fundamental Theorem of Integral Calculus
1335, William of Heytesbury
(Merton College, Oxford)
*Rules for Solving Sophisms*

Defines “instantaneous velocity”

States *Mertonian Rule:*

Under uniform acceleration, the distance traveled is equal to the distance traveled at a constant velocity given by the average of the initial and final velocities.
Nicole Oresme

*Tractatus de configurationaibus qualitatum et motuum*

*(Treatise on the Configuration of Qualities and Motions)*

*Circa 1350*

---

Geometric demonstration of the Mertonian Rule:
1638, Galileo Galilei’s
*The Discourses and Mathematical Demonstrations Relating to Two New Sciences*

velocity = $gt$

distance = $\frac{gt}{2} \cdot t = \frac{1}{2} gt^2$
Gottfried Leibniz: I shall now show that the general problem of quadratures [areas] can be reduced to the finding of a line that has a given law of tangency (declivitas), that is, for which the sides of the characteristic triangle have a given mutual relation. Then I shall show how this line can be described by a motion that I have invented.

Supplementum geometriæ dimensoriae, *Acta Eruditorum*, 1693
Isaac Barrow: “I add one or two theorems, which it will be seen are of great generality, and not lightly to be passed over.”

*Geometrical Lectures*, 1670

DF proportional to area VZGED, FT tangent to VIF at F, then DT is proportional to DF/DE.
James Gregory:

*Geometriae pars universalis (The Universal Part of Geometry), 1668*

Given the equation of a curve, shows how to use integration to compute arc length.

Given the expression for arc length, shows how to find the equation of the corresponding curve. This requires proving that if $u$ represents the area under the curve whose height is $z$, then $z$ is the rate at which $u$ is changing.
Isaac Newton, the October 1666 Tract on Fluxions (unpublished):

“Problem 5: To find the nature of the crooked line [curve] whose area is expressed by any given equation.”

Let \( y \) be the area under the curve \( ac \), then “the motion by which \( y \) increaseth will bee \( bc = q \).”

\[
y = \int_a^b q(b) \, db
\]

\[
\frac{dy}{dt} = q(b) \frac{db}{dt}
\]
Isaac Newton, the October 1666 Tract on Fluxions (unpublished):

“Problem 7: The nature of any crooked line being given to find its area, when it may bee.”

The area under the curve $q = \frac{ax}{\sqrt{a^2 - x^2}}$
is $-a\sqrt{a^2 - x^2}$. 
Isaac Newton, the October 1666 Tract on Fluxions (unpublished):

“Problem 7: The nature of any crooked line being given to find its area, when it may bee.”

The area under the curve

\[ q = \sqrt{\frac{x^3}{a}} - \frac{e^2 b}{x \sqrt{ax - x^2}} \]

is

\[ \frac{2}{5} \sqrt{\frac{x^5}{a}} + \frac{2e^2 b}{ax} \sqrt{ax - x^2}. \]
Isaac Newton, the October 1666 Tract on Fluxions (unpublished):

“Problem 7: The nature of any crooked line being given to find its area, when it may bee.”

If we know that
\[ f(x) = g'(x) \]
then
\[ \int_{a}^{b} f(x) = g(b) - g(a) \]
Until 1823, there was no Fundamental Theorem of Calculus.

Integration was defined as the inverse of differentiation (antidifferentiation).

Newton’s insight was establishing the connection between antidifferentiation and accumulations.
Cauchy, 1823, first explicit definition of definite integral as limit of sum of products

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})(x_{i} - x_{i-1}). \]

Purpose is to show that the definite integral is well-defined for any continuous function.
He now needs to connect this to the antiderivative. Using the mean value theorem for integrals, he proves that
\[
\frac{d}{dt} \int_0^t f(x)\,dx = f(t).
\]

By the mean value theorem, any function whose derivative is 0 must be constant. Therefore, any two functions with the same derivative differ by a constant. Therefore, if \( F \) is any antiderivative of \( f \), then
\[
\int_0^t f(x)\,dx = F(t) + C \quad \Rightarrow \quad \int_a^b f(x)\,dx = F(b) - F(a).
\]
Three Lessons:

1. The real point of the FTIC is that there are two conceptually different but generally equivalent ways of interpreting integration: as antidifferentiation and as a limit of approximating sums.

2. The modern statement of the FTIC is the result of centuries of refinement of the original understanding and requires considerable unpacking if students are to understand and appreciate it.

3. The FTIC arose from a dynamical understanding of total change as an accumulation of small changes proportional to the instantaneous rate of change. This is where we need to begin to develop student understanding of the FTIC.
Problem: Most students conceptualize the definite integral as an area. What does it mean to say that distance is the definite integral of the velocity?

“I don’t understand how a distance can be an area.”

Quote from student of Pat Thompson at Arizona State University
How can we help students understand accumulation as a function?
Sketch the graph of volume as a function of height.

Pat Thompson, ASU:

How can we help students understand the limiting process required for the definition of the accumulation function?
Let $f(x) = x^2 + 1$.

Explain the meaning of $\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$.

“You take your values and you squish them really small until ... you can go no more, and magically that’s the limit.”

(student quoted in Oehrtman, 2011)
The problem: Students understand limits in terms of what is happening to the $h$. Mathematicians understand limits in terms of the values of $(f(3+h) - f(3))/h$.

The solution: Don’t use the language of “limits.” Focus on upper and lower bounds on the expression in question and what can be done to tighten these bounds.
Coherent Labs to Enhance Accessible and Rigorous Calculus Instruction

This project is supported by the National Science Foundation, grant numbers 1245021 & 1245178.

Clearcalculus.okstate.edu

PowerPoint available at www.macalester.edu/~bressoud/talks

Mike Oehrtman