A Little Algebra

A supplement to *The Art of Counting*

David M. Bressoud*

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You can find the PowerPoint presentation of *The Art of Counting* at http://www.macalester.edu/∼bressoud/talks.

Contents

1 Manipulating the plane partition generating function

We let $pp(n)$ represent the number of plane partitions of n, the number of ways of stacking n boxes into a corner. Major MacMahon proved that

$$\begin{aligned} 1+q+3q^2+6q^3+13q^4+\cdots &= 1+\sum_{j=1}^{\infty} pp(j)q^j \\ &= \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} = \frac{1}{(1-q)} \cdot \frac{1}{(1-q^2)^2} \cdot \frac{1}{(1-q^3)^3} \cdots \quad (1) \end{aligned}$$

The proof of this result is not simple and relies on determinant evaluations. Two different proofs can be found in *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*.

*Macalester College, St. Paul, MN; bressoud@macalester.edu; www.macalester.edu/∼bressoud

The key to turning this representation for the generating function into a recursion is the geometric series and a variation of it:

$$\begin{aligned} t^n - 1 &= t^n - t^{n-1} + t^{n-1} - t^{n-2} + t^{n-2} - t^{n-3} = \cdots + t - 1 \\ &= (t-1)t^{n-1} + (t-1)t^{n-2} + (t-1)t^{n-3} + \cdots + (t-1) \\ &= (t-1)(t^{n-1} + t^{n-2} + t^{n-3} + \cdots + 1), \qquad (2) \\ \frac{t^n - 1}{t-1} &= t^{n-1} + t^{n-2} + t^{n-3} + \cdots + 1. \qquad (3) \end{aligned}$$

If we take the limit as t approaches 1 on each side of this equality, we see that

$$\lim_{t\to 1} \frac{t^n - 1}{t-1} = \lim_{t\to 1}(t^{n-1} + t^{n-2} + t^{n-3} + \cdots + 1) = n. \qquad (4)$$

Note that when the numerator and denominator of a fraction *both* approach 0, the limit of the fraction might approach anything, depending on how fast each approaches zero. In this case, the geometric series enables us to find the exact value of this limit.

From equation (1), we see that

$$\frac{\left(1 + \sum_{j=1}^{\infty} pp(j) q^j t^j\right) - \left(1 + \sum_{j=1}^{\infty} pp(j) q^j\right)}{t-1} = \frac{\left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k t^k)^k}\right) - \left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}\right)}{t-1}. \qquad (5)$$

We take the left side of equation (5) and see what happens as t approaches 1:

$$\begin{aligned} \lim_{t\to 1} \frac{\left(1 + \sum_{j=1}^{\infty} pp(j) q^j t^j\right) - \left(1 + \sum_{j=1}^{\infty} pp(j) q^j\right)}{t-1} &= \lim_{t\to 1} \sum_{j=1}^{\infty} pp(j) q^j \frac{t^j - 1}{t-1} \\ &= \sum_{j=1}^{\infty} pp(j)\, j\, q^j \qquad (6) \end{aligned}$$

The limit of the right side of equation (5) will take more work. We begin with a generalization of the idea used to derive the sum of the geometric series, equation (3). Let

$f_1, f_2, \ldots, f_n$ be any n functions, then

$$\begin{aligned}
\prod_{k=1}^{n} f_k(qt) - \prod_{k=1}^{n} f_k(q) &= \prod_{k=1}^{n} f_k(qt) - f_1(q) \prod_{k=2}^{n} f_k(qt) \\
&\quad + f_1(q) \prod_{k=2}^{n} f_k(qt) - f_1(q) f_2(q) \prod_{k=3}^{n} f_k(qt) \\
&\quad + f_1(q) f_2(q) \prod_{k=3}^{n} f_k(qt) - f_1(q) f_2(q) f_3(q) \prod_{k=4}^{n} f_k(qt) \\
&\quad + \cdots + f_1(q) f_2(q) \cdots f_{n-1}(q) f_n(qt) - \prod_{k=1}^{n} f_k(q) \\
&= [f_1(qt) - f_1(q)] \prod_{k=2}^{n} f_k(qt) \\
&\quad + f_1(q) [f_2(qt) - f_2(q)] \prod_{k=3}^{n} f_k(qt) \\
&\quad + f_1(q) f_2(q) [f_3(qt) - f_3(q)] \prod_{k=4}^{n} f_k(qt) \\
&\quad + \cdots + \prod_{k=1}^{n-1} f_k(q) [f_n(qt) - f_n(q)] \\
&= \left[\frac{f_1(qt)}{f_1(q)} - 1\right] f_1(q) \prod_{k=2}^{n} f_k(qt) + \left[\frac{f_2(qt)}{f_2(q)} - 1\right] f_1(q) f_2(q) \prod_{k=3}^{n} f_k(qt) \\
&\quad + \cdots + \left[\frac{f_n(qt)}{f_n(q)} - 1\right] \prod_{k=1}^{n} f_k(q) \qquad (7)
\end{aligned}$$

Now divide by $t-1$ and take the limit as t approaches 1:

$$\lim_{t\to 1} \frac{\prod_{k=1}^{n} f_k(qt) - \prod_{k=1}^{n} f_k(q)}{t-1} = \left(\sum_{k=1}^{n} \lim_{t\to 1} \frac{1}{t-1} \left[\frac{f_k(qt)}{f_k(q)} - 1\right] \right) \prod_{k=1}^{n} f_k(q). \qquad (8)$$

For our problem, we want the upper limit to be infinite. That is not a problem. And we want $f_k(q) = 1/(1-q^k)^k$. We now investigate the limit inside the summation with this value for f_k:

$$\begin{aligned}
\lim_{t\to 1} \frac{1}{t-1}\left[\frac{f_k(qt)}{f_k(q)} - 1\right] &= \lim_{t\to 1} \frac{1}{t-1}\left[\frac{(1-q^k)^k}{(1-q^k t^k)^k} - 1\right] \\
&= \lim_{t\to 1} \frac{\left(\frac{1-q^k}{1-q^k t^k}\right)^k - 1}{\left(\frac{1-q^k}{1-q^k t^k}\right) - 1} \cdot \frac{\frac{1-q^k}{1-q^k t^k} - 1}{t-1} \\
&= \lim_{t\to 1} \frac{\left(\frac{1-q^k}{1-q^k t^k}\right)^k - 1}{\left(\frac{1-q^k}{1-q^k t^k}\right) - 1} \cdot \frac{q^k(t^k-1)}{(1-q^k t^k)(t-1)}. \qquad (9)
\end{aligned}$$

As t approaches 1, the quantity $(1-q^k)/(1-q^k t^k)$ also approaches 1, and therefore the first fraction approaches k. The second fraction approaches $kq^k/(1-q^k)$. We have demonstrated that

$$\lim_{t\to 1} \frac{1}{t-1}\left[\frac{f_k(qt)}{f_k(q)} - 1\right] = \frac{k^2 q^k}{1-q^k}. \qquad (10)$$

We substitute this result in equation (8) with n replaced by ∞, and we get

$$\lim_{t\to 1} \frac{\left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k t^k)^k}\right) - \left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}\right)}{t-1} = \left(\sum_{k=1}^{\infty} \frac{k^2 q^k}{1-q^k}\right) \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}. \qquad (11)$$

The summation on the right side can be further simplified, first by using the infinite geometric series (valid for $|t| < 1$),

$$\frac{t}{1-t} = t + t^2 + t^3 + t^4 + \cdots . \qquad (12)$$

Our summation can be rewritten as

$$\sum_{k=1}^{\infty} \frac{k^2 q^k}{1-q^k} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k^2 q^{lk}. \qquad (13)$$

Let $m = lk$. The variable m takes on every positive integer, and if we first decide on the value for m, then the corresponding values of k are simply the divisors of m (written $k|m$),

$$\sum_{k=1}^{\infty} \frac{k^2 q^k}{1-q^k} = \sum_{m=1}^{\infty} \left(\sum_{k|m} k^2\right) q^m. \qquad (14)$$

The sum of the squares of the divisors of an integer is a common function of number theory, usually written as $\sigma_2(m)$. In *Mathematica*, it is built in as `DivisorSigma[2,m]`.

In equation (11), we use the result we have just found and replace the product by the sum on the left side of equation (1), defining $pp(0) = 1$,

$$\lim_{t\to 1} \frac{\left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k t^k)^k}\right) - \left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}\right)}{t-1} = \sum_{m=1}^{\infty} \sigma_2(m) q^m \sum_{n=0} pp(n) q^n$$

$$= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sigma_2(m)\ pp(n) q^{m+n} \tag{15}$$

Again, we combine our indices. Let $j = m + n$, then m goes from 1 to j, and we can rewrite our limit as

$$\lim_{t\to 1} \frac{\left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k t^k)^k}\right) - \left(\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}\right)}{t-1} = \sum_{j=1}^{\infty} \left(\sum_{m=1}^{j} \sigma_2(m)\ pp(j-m) \right) q^j. \tag{16}$$

Looking back at equations (5) and (6), we see that

$$\sum_{j=1}^{\infty} pp(j)\, j\, q^j = \sum_{j=1}^{\infty} \left(\sum_{m=1}^{j} \sigma_2(m)\ pp(j-m) \right) q^j. \tag{17}$$

The coefficients of q^j must be the same, and so we get our recursion:

$$pp(j) = \frac{1}{j} \sum_{m=1}^{j} \sigma_2(m)\ pp(j-m). \tag{18}$$

2 The formula for the number of alternating sign matrices

In this section, we will see how to find the formula for A_n, the number of alternating sign matrices, using the conjectured formula for ratio of consecutive terms in the Pascal-like triangle of values of $A_{n,k}$,

$$\frac{A_{n,k+1}}{A_{n,k}} = \frac{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}{\binom{n-2}{k-1} + \binom{n-1}{k-1}}, \tag{19}$$

and the fact that

$$A_{n,1} = \sum_{k=1}^{n-1} A_{n-1,k} = A_{n-1}. \tag{20}$$

We first express A_n in terms of A_{n-1} and the ratios:

$$\begin{aligned} A_n &= A_{n,1} + A_{n,2} + A_{n,3} + \cdots + A_{n,n} \\ &= A_{n,1}\left(1 + \frac{A_{n,2}}{A_{n,1}} + \frac{A_{n,2}}{A_{n,1}} \cdot \frac{A_{n,3}}{A_{n,2}} + \cdots + \frac{A_{n,2}}{A_{n,1}} \cdots \frac{A_{n,n}}{A_{n,n-1}}\right) \\ &= A_{n-1}\left(1 + \frac{A_{n,2}}{A_{n,1}} + \frac{A_{n,2}}{A_{n,1}} \cdot \frac{A_{n,3}}{A_{n,2}} + \cdots + \frac{A_{n,2}}{A_{n,1}} \cdots \frac{A_{n,n}}{A_{n,n-1}}\right). \end{aligned} \tag{21}$$

Our next step is to simplify the ratio $A_{n,k+1}/A_{n,k}$,

$$\begin{aligned} \frac{A_{n,k+1}}{A_{n,k}} &= \frac{\binom{n-2}{n-k-1} + \binom{n-1}{n-k-1}}{\binom{n-2}{k-1} + \binom{n-1}{k-1}} \\ &= \frac{\frac{(n-2)!}{(n-k-1)!(k-1)!} + \frac{(n-1)!}{(n-k-1)!k!}}{\frac{(n-2)!}{(n-k-1)!(k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}} \\ &= \frac{\frac{(n-2)!}{(n-k-1)!k!}(k+n-1)}{\frac{(n-2)!}{(n-k)!(k-1)!}(n-k+n-1)} \\ &= \frac{(n-k)(n+k-1)}{k(2n-k-1)}. \end{aligned} \tag{22}$$

Now we simplify the product of ratios,

$$\begin{aligned} \prod_{k=1}^{t} \frac{A_{n,k+1}}{A_{n,k}} &= \frac{(n-1)(n-2)\cdots(n-t)\cdot n(n+1)\cdots(n+t-1)}{1\cdot 2\cdots t\cdot(2n-2)(2n-3)\cdots(2n-t-1)} \\ &= \frac{(n-t-1)!(2n-t-2)!}{t!\,(n-t-1)!\,(2n-2)!} \\ &= \frac{(n-1)!^2}{(2n-2)!} \cdot \frac{(n-t-1)!}{t!\,(n-1)!} \cdot \frac{(2n-t-2)!}{(2n-t-1)!(n-1)!} \\ &= \frac{(n-1)!^2}{(2n-2)!} \cdot \binom{n+t-1}{n-1} \cdot \binom{2n-t-2}{n-1} \end{aligned} \tag{23}$$

Combining this with equation (21), we have that

$$A_n = A_{n-1} \sum_{t=0}^{n-1} \frac{(n-1)!^2}{(2n-2)!} \cdot \binom{n+t-1}{n-1} \cdot \binom{2n-t-2}{n-1}. \tag{24}$$

We next prove that

$$\binom{3n-2}{2n-1} = \sum_{t=0}^{n} \binom{n+t-1}{n-1} \cdot \binom{2n-t-2}{n-1}. \tag{25}$$

The left side of this equation counts the number of ways of choosing $2n-1$ objects from a set of $3n-2$. Let the set from which we are choosing consist of the integers $\{1, 2, 3, \ldots, 3n-2\}$. What is the n^{th} integer that we choose? It cannot be smaller than n or larger than $2n-1$ (because we still have to choose $n-1$ more integers). Let the nth integer that we choose be $n+t$ where $0 \le t \le n-1$. For this choice of t, we still need to choose $n-1$ integers from first $n+t-1$ and $n-1$ integers from the last $2n-t-2$. That is precisely what the t^{th} summand counts.

We now use equation (25) to finish the formula for A_n:

$$\begin{aligned} A_n &= A_{n-1} \frac{(n-1)!^2}{(2n-2)!} \binom{3n-2}{2n-1} \\ &= A_{n-1} \frac{(n-1)!^2 \, (3n-2)!}{(2n-2)! \, (2n-1)! \, (n-1)!} \\ &= A_{n-1} \frac{(n-1)! \, (3n-2)!}{(2n-2)! \, (2n-1)!}. \end{aligned} \tag{26}$$

We know that $A_1 = 1$, and so

$$\begin{aligned} A_n &= \frac{1! \cdot 4!}{2! \cdot 3!} \cdot \frac{2! \cdot 7!}{4! \cdot 5!} \cdot \frac{3! \cdot 10!}{6! \cdot 7!} \cdots \frac{(n-1)! \, (3n-2)!}{(2n-2)! \, (2n-1)!} \\ &= \frac{1! \cdot 2! \cdots (n-1)! \; \cdot \; 4! \cdot 7! \cdots (3n-2)!}{1! \cdot 2! \cdot 3! \cdots (2n-1)!} \\ &= \frac{4! \cdot 7! \cdots (3n-2)!}{n! \cdot (n+1)! \cdots (2n-1)!} \\ &= \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \end{aligned} \tag{27}$$

And there it is!

3 Further Questions

Questions involving the number of regions of space cut by k planes:

1. The formula for the number of regions in space cut by k planes can be represented as

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \binom{k}{3}.$$

Can you explain directly why the regions should equal the number of ways of choosing none of the planes, plus the number of ways of choosing one of the planes, plus the number of ways of choosing two of the planes, plus the number of ways of choosing three of the planes?

2. Can you find and *prove* the formula for the number of regions of 4-dimensional hyperspace cut by k 3-dimensional hyperplanes? What assumptions do you need to make about 3-dimensional hyperplanes in 4-dimensional hyperspace?

3. Can you find and *prove* the formula for the number of regions of n-dimensional hyperspace cut by k $(n-1)$-dimensional hyperplanes? What assumptions do you need to make about $(n-1)$-dimensional hyperplanes in n-dimensional hyperspace?

4. Find and *prove* the formula for the number of finite regions in space cut by k planes.

Questions involving partitions and plane partitions:

1. A **partition** of n is a representation of n as a sum of positive integers (order does not matter). Thus we have five partitions of 4:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad \text{and } 1+1+1+1.$$

The generating function for partitions, $p(n)$, is given by

$$1+\sum_{n=1}^{\infty} p(n)\, q^n = \frac{1}{(1-q)}\,\frac{1}{(1-q^2)}\,\frac{1}{(1-q^3)}\,\frac{1}{(1-q^4)}\cdots. \tag{28}$$

Use this generating function to find a recursive formula for $p(n)$, the number of partitions of n.

2. In the 1700s, Euler[1] discovered an even better recursive formula for $p(n)$. Start expanding the reciprocal of the generating function until you can guess what this product looks like as a power series:

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots = 1-q-q^2+\cdots.$$

Now use the fact that

$$\left(1+\sum_{n=1}^{\infty} p(n)\, q^n\right)\left(1-q-q^2+\cdots.\right)$$

to find a very efficient recursion for finding the values of $p(n)$.

[1] pronouced "Oyler"

3. A **combination** of n is a representations as a sum of positive integers in which the order does matter. Thus we have seven combinations of 4:

   $$4, \quad 3+1, \quad 1+3, \quad 2+2, \quad 2+1+1, \quad 1+2+1, \quad 1+1+2, \quad \text{and } 1+1+1+1.$$

   The formula for the number of combinations of n into at most k parts can be represented by a certain binomial coefficient. Find this formula.

4. A **symmetric plane partition** is a plane partition that is symmetric about the main diagonal. For example,

   $$\begin{array}{cccc} 4 & 4 & 2 & 1 \\ 4 & 3 & 1 & \\ 2 & 1 & 1 & \\ 1 & & & \end{array} \quad , \quad \text{and} \quad \begin{array}{ccccc} 5 & 4 & 3 & 3 & 1 \\ 4 & 3 & 3 & 2 & \\ 3 & 3 & 1 & & \\ 3 & 2 & & & \\ 1 & & & & \end{array}$$

   are symmetric plane partitions. The generating function for symmetric plane partitions is given by

   $$\begin{aligned} 1+\sum_{n=1}^{\infty} spp(n)\, q^n &= \frac{1}{(1-q)(1-q^2)} \frac{1}{(1-q^3)(1-q^4)^2} \frac{1}{(1-q^5)(1-q^6)^3} \cdots \\ &= \prod_{k=1}^{\infty} \frac{1}{(1-q^{2k-1})(1-q^{2k})^k} \cdots . \end{aligned} \tag{29}$$

   Use this generating function to find a recursive formula for $spp(n)$, the number of symmetric plane partitions of n.

Questions involving alternating sign matrices:

1. Let $\mathcal{A}_n$ be the set of all $n \times n$ alternating sign matrices and for $A \in \mathcal{A}_n$, let $N(A)$ be the number of -1s in A. Define the polynomials

   $$A(n;x) = \sum_{A \in \mathcal{A}_n} x^{N(A)}.$$

   Show that

   $$\begin{aligned} A(1;x) &= 1, \\ A(2;x) &= 2, \\ A(3;x) &= 6+x, \\ A(4;x) &= 24+16x+2x^2, \\ A(5;x) &= 120+200x+94x^2+14x^3+x^4, \\ A(6;x) &= 720+2400x+2684x^2+1284x^3+310x^4+36x^5+2x^6, \\ A(7;x) &= 5040+29400x+63308x^2+66158x^3+38390x^4+13037x^5 \\ &\quad + 2660x^6+328x^7+26x^8+x^9. \end{aligned}$$

You can find a *Mathematica* program for generating these polynomials on page 91 of *Proofs and Confirmations*.

2. Explain why $A(n; 0) = n!$.
3. Find a formula for the degree of $A(n; x)$ (degree is the highest power of x), and then prove that your formula is correct.
4. Guess a formula for $A(n; 2)$.
5. Guess a formula for $A(n; 3)$.

4 References

1. George Andrews and Kimmo Eriksson, *Integer Partitions*, Cambridge University Press, 2005

   This is the best elementary introduction to partitions.

2. George Andrews, *The Theory of Partitions*, Cambridge University Press, 1984.

   This is the most comprehensive resource on partitions.

3. David Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, Mathematical Association of America and Cambridge University Press, 1999.

   The full story up until 1999 of the alternating sign matrix conjecture.

4. George Pólya , *Let Us Teach Guessing* (video), Mathematical Association of America.

   This video, filmed at Stanford in 1965, introduces the problem of the number of regions in space cut by k planes.