Centers of Mass

adapted from *Calculus: Modeling and Application*
by David A. Smith and Lawrence C. Moore

For this project, you will be working in groups of 3 or 4 and writing up your solutions in groups of 2 or 3. Before you begin, choose one person to be the recorder and one to be the prompter. The duty of the **recorder** is to write down the ideas that you try and the insights and solutions that you discover. The job of the **prompter** is to make sure that everyone gets a chance to contribute and that everyone understands what has been done. Never hesitate to say if you don't quite understand what has been said or done. It is when one person explains what they think they understand to someone else that uncertainties or weaknesses can be found.

The answers to this project are to be submitted as a project report. It should be possible to find answers to all of the questions by reading your report, but do **not** lay out the report in a question-answer format. It should be written as a report to someone who asked you to figure out how to calculate the moments of the four cutouts. It should be written so that a student in another calculus class could pick it up and understand what you have done and why you have done it. You should not assume that the person who reads your report knows what the questions are. Start with an introduction that explains the problem and lets the reader know what you have found. Explain how you got to this solution, how confident you are, and why you are or are not confident of this answer. Write clearly using complete sentences and well-structured paragraphs. There should be a brief conclusion summarizing what you accomplished in this report.

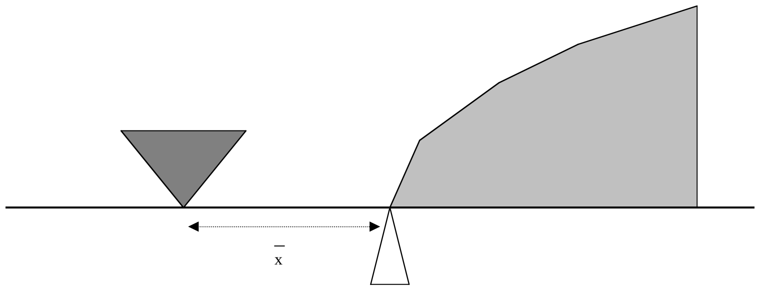
The first draft will not be graded. I will comment on it and return it to you for improvements. I am looking not just for correct answers, but also for clear explanations.

1. Record the x and y coordinates of the balance points of your plywood cutouts. All of the cutouts lie in the unit square: The lengths of the bottom and right-hand edges are one unit. These coordinates mark the center of mass and are denoted by $\overline{x}$ and $\overline{y}$. The "bar" notation stands for "average." We are locating the average point for the distribution of mass.

	Bounding curve	measured $\overline{x}$	measured $\overline{y}$
A.			
B.			
C.			
D.			

The center of mass is a point at which, if we concentrated all the mass at that one point, the object would have the same behavior on a balance beam. Image that we place one of our cutouts on the right-hand side of a balance beam, with the origin at the balance point as in the figure below. To achieve balance by placing the same mass on the left at a single point, that point would have to be $\overline{x}$ units away from the origin.

The contribution of each mass on the balance beam is called a **moment**; in the case of the figure, we speak of "moments with respect to the y-axis," since we are balancing on opposite sides of the line $x = 0$, ie the y-axis. When working with a coordinate system, it is convenient to let *signs* of coordinates



do the work of keeping track of left and right. Thus, we represent the moment of a point mass by the product of the mass and its *directed* distance from the balance axis. Then the condition for balancing is that the **the sum of the moments is zero**.

For the figure, the moment of the cutout with respect to the y-axis is unknown, so we just give it a name: M_y. The moment of the point mass on the left is $-m\overline{x}$. The balancing condition is

$$M_y - m\,\overline{x} = 0. \tag{1}$$

2. Solve equation (1) for $\overline{x}$ in terms of the unknown quantities M_y and m. The problem of finding $\overline{x}$ is now reduced to that of finding M_y and m.
3. Each of your cutouts has a constant thickness (let's call it θ) and a constant density (let's call it δ). If, in the course of this calculation, you can demonstrate a need to know either of those numbers, I will tell you what they are. Otherwise, just refer to them by these names. Explain briefly why the problem of finding m can be reduced to the problem of finding the area of your cutout, once you know θ and δ.
4. Calculate the areas and masses of all of your cutouts and record the results in the table.

	Bounding curve	Area	Mass
A.			
B.			
C.			
D.			

5. Now we tackle the harder problem of finding M_y. Sketch one of your four shapes, including the coordinate axes in your sketch. Divide your shape into eight vertical strips of equal width. For the third strip from the left, estimate the area, the mass, the coordinates of the center of mass, and the moment with respect to the y-axis.

   If you can estimate M_y for a single nearly-rectangular strip, then you can estimate M_y for the entire cutout by adding up the contributions from all the strips. (Recall the additive behavior of multiple masses on the balance beam.) *Doing* this kind of estimation is extremely tedious, but *imagining* it is easy, and that leads to an easy way to actually calculate M_y.
6. Suppose that you have subdivided your shape into n strips. (In the previous question, n was 8.) Let Δx represent the width of each strip. If the interval you subdivided runs from $x = a$ to $x = b$, how is Δx related to a, b, and n?

Suppose you number the strips 1, 2, 3, …, n and then label the points of subdivision on the x-axis (including the end points a and b) by x_0 (which is a), x_1, x_2, …, x_{n-1}, x_n (which is b). Let k be the index variable that takes the values 1, 2, 3, …, n; then the kth strip lies between x_{k-1} and x_k. (In the previous section, you looked at the strip for $k = 3$.) Find x_k and x_{k-1} in terms of k, a, and Δx.

Let's denote the midpoint of the kth subinterval by $\overline{x}_k$. Write an expression for $\overline{x}_k$ in terms of x_{k-1} and x_k. Then write another expression for $\overline{x}_k$ in terms of k, a and Δx.

7. Approximate the kth strip by a rectangle whose height is $f(\overline{x}_k)$.
   (a) What are the coordinates of the center of mass of this rectangle?
   (b) What is the area of this rectangle?
   (c) What is the moment of this rectangle with respect to the y-axis?
   (d) Write out the sum of the moments with respect to the y-axis for all n rectangles.
8. The sum in question 7 (d) is an approximating sum for some integral: As n gets larger and the strips get smaller, the approximations—both to the exact moment and to some integral—get better. It is therefore reasonable to suppose that the integral being approximated *is* the moment we seek. What integral is being approximated by the sum in question 7 (d)? Your answer to this question is a formula for M_y.
9. Check your formula for M_y by evaluating it for one of your shapes (pick the easiest one) and using your formula from question 2 and mass from question 4 to find $\overline{x}$. Is the answer close to the measured value of $\overline{x}$ in question 1? (If not, you may need to rethink something at this point.)
10. It is time to think about how to calculate $\overline{y}$. For that purpose, we need to think about balancing about the x-axis. You can choose whether to mentally rotate your figures 90°, physically rotate them 90°, or imagine gravity acting sidewise. Reread the two paragraphs before question 2 and convince yourself that the moment of the figure with respect to the x-axis, M_x, must satisfy the condition

$$M_x - m\,\overline{y} = 0. \tag{2}$$

11. Find a formula for $\overline{y}$ in terms of m and M_x. The problem of finding $\overline{y}$ is now reduced to that finding two other quantities, one of which you already know.
12. You have already done most of the hard work to find a formula for M_x. You can use the same subdivision of the area into approximately rectangular strips, calculate the moment for each strip, add the results, and determine what integral is being approximated by the sum. But be careful—for M_y the balance axis was parallel to the strips. For M_x it is perpendicular, so there is no reason to think that the answer will have the same form.

    Use your results from question 7 to find the moment with respect to the x-axis of the kth rectangular strip. Then sum the moments of the individual strips to find an expression for approximating M_x.
13. The sum in question 11 is an approximating sum for some integral: As n gets larger and the strips get smaller, the approximations—both to the exact moment and to some integral—get better. It is therefore reasonable to suppose that the integral being approximated *is* the moment we seek. What integral

is being approximated the sum in question 11? Your answer to this question is a formula for M_x.

14. Check your formula for M_x by evaluating it for one of your shapes (pick the easiest one) and then using your formula from question 10 and your mass from question 4 to find $\overline{y}$. Is the answer close to your measured $\overline{y}$ in question 1? (If not, you may need to rethink something at this point.)
15. Fill in the following table. Write a sentence or two about how close your measured centers of mass (from question 1) are to the ones you calculate by integral formulas for this table.

	Bounding curve	Mass	M_y	M_x	$\overline{x}$	$\overline{y}$
A.						
B.						
C.						
D.						