Reliability Theory

adapted from *Calculus: Modeling and Application*
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An Improbably Scenario

Shortly after his inauguration, President Shrub decided to stage a media event in the White House Garden. He would have a large display panel made up, with a thousand light bulbs and, in large block letters across the top, the slogan "A Thousand Points of Light." He would make a short speech and then, in a dramatic gesture, throw a switch to light the bulbs and emphasize his plan for volunteerism.

It fell to a long-time civil servant, Arthur Carewell, to arrange the ceremony. He contracted with carpenters to make the panel, with painters for the sign, with electricians to install the sockets for the bulbs, and with gardeners to erect the sign in the Rose Garden. Being a naturally cautious soul, Arthur ordered 2000 bulbs and had half of them screwed into the sockets; the remainder he placed in a White House storeroom. Being a naturally frugal soul, Arthur bought the cheapest bulbs he could find.

So it came to pass that, at 5:00 one early January evening, the ceremony took place. The president threw the switch, all the bulbs lit up in a dazzling blaze, and the event got at least 15 seconds on all the evening news shows. The lighted display was then forgotten by everyone except Arthur. What should he do now? He was reluctant to tear the display down; the president might want to show it to a visitor. Who was he, a lowly civil servant, to turn off what the president himself had brought to life?

At 5:00 the next evening, Arthur checked the board and noted that some of the bulbs had burned out, 70 to be exact. He recorded the number. Every day at 5:00 he came by to check the board, waiting for the day when all the bulbs would burn out and he could take the display down. Every day he wrote down the number of bulbs burned out. We record some of his data on the next page.

Day	# of bulbs burned out	Fraction of bulbs burned out
0	0	
1	70	
2	142	
3	207	
5	320	
7	411	
14	656	
21	798	
28	882	
35	929	
42	960	
49	978	

Fill in the last column of this table.

1. Let $F(t)$ be the fraction of bulbs burned out as a function of time measured in days. Graph $F(t)$. Explain why

$$F(0) = 0 \qquad \text{and} \qquad \lim_{t \to \infty} F(t) = 1.$$

2. Explain why $F(b) - F(a)$ is the probability that a bulb will burn out between days a and b.

3. Explain why $F(t)$ should satisfy a differential equation of the form:

$$\frac{dF}{dt} = k(1 - F).$$

   Solve this equation using the initial condition $F(0) = 0$ to find F as a function of the variable t with parameter k.

4. Use the data given in the table to determine a good approximation for k. Explain how you got k and why you know that your estimate is a good one. Why can we assume that all of the lightbulbs have burned out by the end of 91 days?

   Interlude: discrete expected lifetimes

5. Suppose we have 100 type A light bulbs that each burn out in exactly 21 days and 200 type B light bulbs that each burn out in exactly 35 days. What is the average lifetime of this collection of 300 light bulbs? This average lifetime is also known as the **expected** lifetime.

6. Suppose that we have a collection of lightbulbs of three types: type I has a lifetime of exactly 23 days, type II has a lifetime of exactly 35 days,

and type II has a lifetime of exactly 42 days. We also know that if we choose a lightbulb at random the probability of choosing type I is 1/4, of choosing type II is 1/2, and of choosing type III is 1/4. What is the expected lifetime of our collection of lightbulbs?

7. Explain why it is that if we have n types of lightbulbs, if type k lasts exactly t_k days, and if we have probability p_k of choosing a lightbulb of type k, then the expected lifetime is

$$p_1 t_1 + p_2 t_2 + \cdots + p_n t_n = \sum_{k=1}^{n} p_k t_k.$$

Back to the real problem

8. Divide the 91 day maximum lifetime of the lightbulbs into m equal intervals and let $\Delta t = 91/m$. Let $t_0, t_1, \ldots, t_m$ be the time intervals Δt apart. Explain why the expected lifetime of a randomly chosen lightbulb can be approximated by

$$\sum_{k=1}^{m} \left[F(t_k) - F(t_{k-1})\right] t_k.$$

9. We would like to let Δt approach 0 and turn this into an integral, but there is no Δt in the summation. Rewrite the approximation as

$$\sum_{k=1}^{m} \left[\frac{F(t_k) - F(t_{k-1})}{\Delta t}\right] t_k \, \Delta t.$$

What integral does this become when you take the limit as $\Delta t \to 0$? What is the expected lifetime of Arthur Carewell's lightbulbs?

Write your report so that it can be read by a calculus student from the other section who has not seen these questions. Do not refer to the questions by number. Your report should have an introduction that explains the problem that you will solve. It should include an explanation of what you did and why you did it and what you discovered. Include the errors you made and how you knew that they were wrong and how you corrected them. Conclude with a summary of what you learned from this project.