More problems on the harmonic series

Problems in case of need.

1. (Infinitude of primes.) Euler used the divergence of the harmonic series to deduce the infinitude of primes, beginning with the "equation"

$$\sum_{k=1}^{\infty} \frac{1}{k} = \prod_{\text{primes } p} \frac{1}{1 - 1/p}.$$

   Assume that the number of primes is finite, and deduce a contradiction.

2. (Summing prime reciprocals.) To show that the sum of the prime reciprocals diverges, Euler took logs of both sides of the equation above, used the Taylor series for the log, and then went to town. Here's a simplified version:

$$\begin{aligned}\log\left(\sum_{k=1}^{\infty}\frac{1}{k}\right) &= \log\left(\prod_{p}\frac{1}{1-1/p}\right) = \sum_{p} -\log\left(1-p^{-1}\right)\\ &= \sum_{p}\left(\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\dots\right) = \sum_{p}\frac{1}{p}+\sum_{p}\frac{1}{p^2}\left(\frac{1}{2}+\frac{1}{3p}+\frac{1}{4p^2}+\dots\right)\\ &< \sum_{p}\frac{1}{p}+\sum_{p}\frac{1}{p^2}\frac{1}{1-p} = \sum_{p}\frac{1}{p}+C,\end{aligned}$$

   for some finite constant C. Since the first quantity is clearly infinite, so must be the sum at the very end.

   Euler concluded, moreover, that since the harmonic partial sums grow like $\log n$, we must have the "equation"

$$\sum_{p}\frac{1}{p} = \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots\right) = \log\log\infty,$$

   which should presumably be interpreted asymptotically.

3. (Knock-out harmonic series.) Omitting some terms from the harmonic series should, presumably, "help" the series converge. Omitting all terms with *composite* denominators isn't enough, as Euler showed (see above). Which of the following "knock-out harmonic series" converges?

   (a) The series formed by omitting all terms whose denominator includes the digit 9.

   (b) The series formed by omitting all terms whose denominator includes the digit string 123456789.