Using the History of Calculus to Enrich our Teaching

assignment for Sunday, August 5, based on "The Pendulum"

Toeplitz describes the analysis of the motion of a pendulum on pages 138144. The pendulum problem is a rich source of mathematics. In the 19th century, it would lead to an intensive study of elliptic integrals, opening new fields for study. Toeplitz's explanation of the development of the basic mathematical model of a pendulum is complete and clear, but he writes at a fairly high level that requires the reader to stop and think about why a given statement is true and what it means in the context of the unfolding argument. This is precisely the kind of reading that we want our students to be able to do.

The purpose of this exercise is to think about what kind of support students would need if challenged to read this passage for understanding. One way of providing this is a series of question that must be answered alongside the reading. A list of some of the possible questions is given below. Your assignment for Sunday is to think about where your students would have difficulty in understanding this passage, and to think about the questions or supplementary material that might assist them. We'll take about half an hour of Sunday's session for you to talk with others in small groups about how you might use this passage in your own class, and will then share these in a general discussion.

1. Why is the acceleration of the end of the pendulum equal to $g\cos\alpha$? What is the relationship between α and ϕ? Students are probably used to seeing 9.8 m/s^2 as the acceleration due to gravity. Why do you think Toeplitz instead uses 10 m/s^2? Think about this last question again after completing the reading of the entire section.

2. What is $\dfrac{d^2\phi}{dt^2}$ and why do we multiply it by $-l$ to get the acceleration of the end of the pendulum: Why the negative? Why does acceleration depend on the length of the pendulum?

3. We are told to multiply both sides of the differential equation by ϕ'. What if $\phi' = 0$? If this is true for all time, what does that say about the pendulum? Describe what is happening to any pendulum at the moment that $\phi'(t) = 0$.

4. Find a differential equation that you could completely solve by using this trick of multiplying both sides by ϕ'.

5. Explain how we get from the original differential equation to the relationship

   $$\frac{l}{2}\left(\phi'\right)^2 = g\cos\phi + C.$$

   Why is C equal to $-g\cos\beta$?

6. Explain how we got from $(l/2)\left(\phi'\right)^2 = g\left(\cos\phi - \cos\beta\right)$ to

$$t = \sqrt{\frac{l}{2g}} \int \frac{d\phi}{\sqrt{\cos\phi - \cos\beta}}.$$

What are the limits on this integral if we want to find the time it takes the pendulum to go from its furthest position to the left to its furthest position to the right? Explain how we got from this integral to

$$t = \sqrt{\frac{l}{g}} \int \frac{du}{\sqrt{1-u^2}\,\sqrt{1-u^2\sin^2(\beta/2)}}.$$

What are the limits on this integral if we want to find the time it takes the pendulum to go from its furthest position to the left to its furthest position to the right?

7. Using the assumption that for $\beta < 10^\circ$, we have that

$$\frac{99}{100} \leq 1 - u^2\sin^2(\beta/2) \leq 1,$$

and ignoring the effects of friction and air resistance, find bounds on the true value of the time it takes to complete one swing from the far left to the far right. Is $2\pi\sqrt{l/g}$ an upper bound or a lower bound on this time?