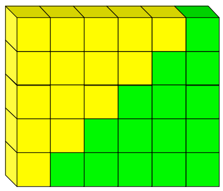
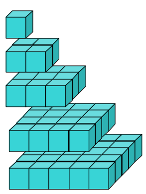
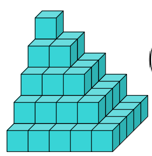


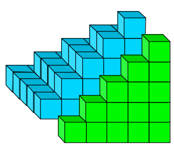
$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

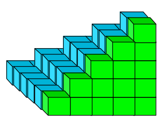
$$(1+2+3+\cdots+n)+(1+2+3+\cdots+n)$$
$$=n(n+1)\ .$$



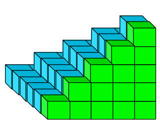


$$\left(1^2+2^2+\cdots+n^2\right)$$

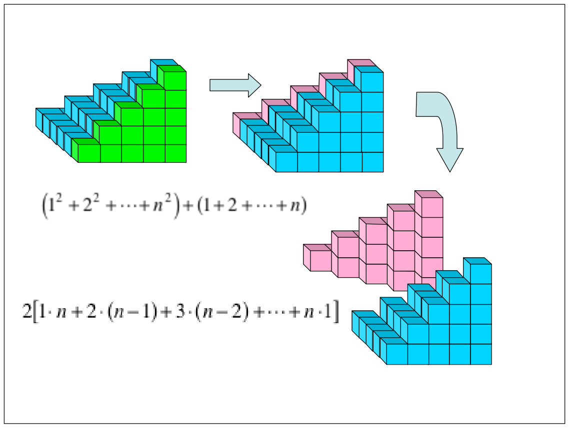


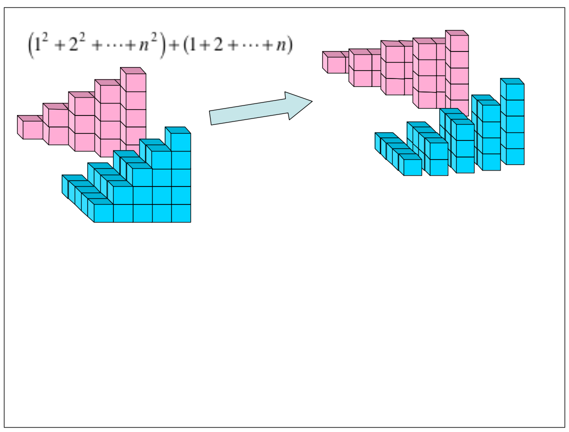


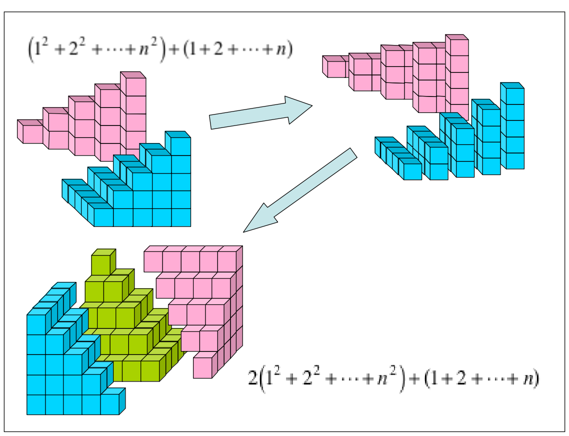
$$\left(1^2+2^2+\cdots+n^2\right)+(1+2+\cdots+n)$$



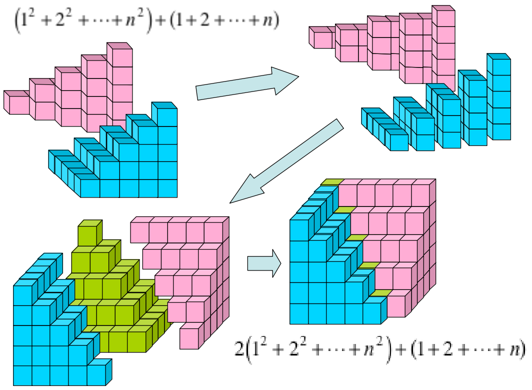
$$\left(1^2+2^2+\cdots+n^2\right)+(1+2+\cdots+n)$$


$(1^2+2^2+\cdots+n^2)+(1+2+\cdots+n)$
$2[1\cdot n+2\cdot(n-1)+3\cdot(n-2)+\cdots+n\cdot 1]$

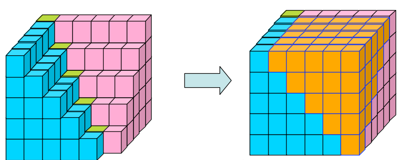

$(1^2+2^2+\cdots+n^2)+(1+2+\cdots+n)$


$(1^2+2^2+\cdots+n^2)+(1+2+\cdots+n)$
$2(1^2+2^2+\cdots+n^2)+(1+2+\cdots+n)$

$\left(1^2+2^2+\cdots+n^2\right)+(1+2+\cdots+n)$



$2\left(1^2+2^2+\cdots+n^2\right)+(1+2+\cdots+n)$



$$3\left(1^2+2^2+\cdots+n^2\right)+(1+2+\cdots+n)$$
$$=n(n+1)^2\ .$$
$$1^2+2^2+\cdots+n^2=\frac{n(n+1)^2}{3}-\frac{n(n+1)}{6}$$
$$=n(n+1)\left(\frac{2n+1}{6}\right).$$

Sums of cubes

$1^3=1,\quad 1^3+2^3=9,\quad 1^3+2^3+3^3=36,\quad 1^3+2^3+3^3+4^3=100,\quad 1^3+\cdots+5^3=225$

$1^2,\quad 3^2=(1+2)^2,\quad 6^2=(1+2+3)^2,\quad 10^2=(1+2+3+4)^2,\quad 15^2=(1+\cdots+5)^2$

Sums of cubes

$1^3 = 1,\quad 1^3 + 2^3 = 9,\quad 1^3 + 2^3 + 3^3 = 36,\quad 1^3 + 2^3 + 3^3 + 4^3 = 100,\quad 1^3 + \cdots + 5^3 = 225$

$1^2,\quad 3^2 = (1+2)^2,\quad 6^2 = (1+2+3)^2,\quad 10^2 = (1+2+3+4)^2,\quad 15^2 = (1+\cdots+5)^2$

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

$1^4 + 2^4 + 3^4 + \cdots + n^4 = ?$

$1^4 = 1;\quad 1^4 + 2^4 = 17;\quad 1^4 + 2^4 + 3^4 = 98;$

$1^4 + 2^4 + 3^4 + 4^4 = 354;$

$1^4 + 2^4 + 3^4 + 4^4 + 5^4 = 979;$

$1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 = 2275;\quad \ldots$

$$\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1 + x \\
(1+x)^2 &= 1 + 2x + x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5
\end{aligned}$$

$$
\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1+x \\
(1+x)^2 &= 1+2x+x^2 \\
(1+x)^3 &= 1+3x+3x^2+x^3 \\
(1+x)^4 &= 1+4x+6x^2+4x^3+x^4 \\
(1+x)^5 &= 1+5x+10x^2+10x^3+5x^4+x^5
\end{aligned}
$$

$$
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(1+x)^4 &= 1+4x+6x^2+4x^3+x^4 \\
(1+x)^5 &= 1+5x+10x^2+10x^3+5x^4+x^5
\end{aligned}
$$

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

$$\binom{1}{k}+\binom{2}{k}+\cdots+\binom{k-1}{k}+\binom{k}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

$$0 + 0 + \cdots + 0$$

$$\binom{1}{k}+\binom{2}{k}+\cdots+\binom{k-1}{k}+\binom{k}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

$$0 \;+\; 0 \;+\cdots+\; 0$$

$$\binom{j}{k}=P_k(j)=\frac{1}{k!}j(j-1)(j-2)\cdots(j-k+1)$$

$$\binom{1}{k}+\binom{2}{k}+\cdots+\binom{k-1}{k}+\binom{k}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

$$0 \;+\; 0 \;+\cdots+\; 0$$

$$\binom{j}{k}=P_k(j)=\frac{1}{k!}j(j-1)(j-2)\cdots(j-k+1)$$

$$P_k(1)+P_k(2)+P_k(3)+\cdots+P_k(n)=P_{k+1}(n+1)$$

$$\begin{aligned}
P_1(j) &= j \\
P_2(j) &= \frac{1}{2}j(j-1)=\frac{j^2}{2}-\frac{j}{2} \\
P_3(j) &= \frac{1}{6}j(j-1)(j-2)=\frac{j^3}{6}-\frac{j^2}{2}+\frac{j}{3} \\
P_4(j) &= \frac{1}{24}j(j-1)(j-2)(j-3)=\frac{j^4}{24}-\frac{j^3}{4}+\frac{11j^2}{24}-\frac{j}{4}
\end{aligned}$$

$$
\begin{aligned}
P_1(j) &= j \\
P_2(j) &= \frac{1}{2} j(j-1) = \frac{j^2}{2} - \frac{j}{2} \\
P_3(j) &= \frac{1}{6} j(j-1)(j-2) = \frac{j^3}{6} - \frac{j^2}{2} + \frac{j}{3} \\
P_4(j) &= \frac{1}{24} j(j-1)(j-2)(j-3) = \frac{j^4}{24} - \frac{j^3}{4} + \frac{11j^2}{24} - \frac{j}{4}
\end{aligned}
$$

$$
\begin{aligned}
j &= P_1(j) \\
j^2 &= 2P_2(j) + P_1(j) \\
j^3 &= 6P_3(j) + 6P_2(j) + P_1(j) \\
j^4 &= 24P_4(j) + 36P_3(j) + 14P_2(j) + P_1(j)
\end{aligned}
$$

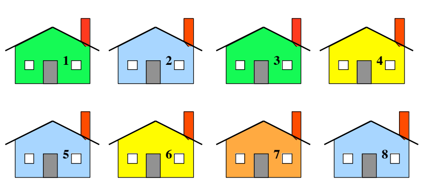
$$
\begin{aligned}
j &= P_1(j) \\
j^2 &= 2P_2(j) + P_1(j) \\
j^3 &= 6P_3(j) + 6P_2(j) + P_1(j) \\
j^4 &= 24P_4(j) + 36P_3(j) + 14P_2(j) + P_1(j) \\
&\vdots
\end{aligned}
$$

All the coefficients are positive integers.

Can you find a simple way of generating them?

Can you discover what they count?

$HP(k,i)$ is the ***House-Painting*** number



It is the number of ways of painting k houses using ***exactly*** i colors.

$$j^k = HP(k,k)\binom{j}{k} + HP(k,k-1)\binom{j}{k-1} +$$
$$+ HP(k,k-2)\binom{j}{k-2} + \cdots + HP(k,1)\binom{j}{1}$$

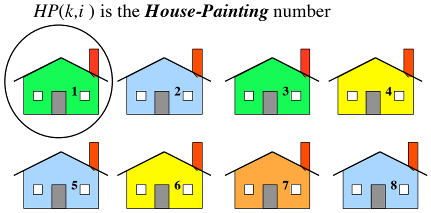
j^k is the number of ways of painting k houses when we have j colors to choose from at each house, and we don' t care whether or not all the colors are used.

1			
2	1		
6	6	1	
24	36	14	1

$HP(k,k) = k!$

$HP(k,1) = 1$

$HP(k,i)$ is the ***House-Painting*** number



$$HP(k,i) = i\left[HP(k-1,i) + HP(k-1,i-1)\right]$$

1

2 1

6 6 1

24 36 14 1

↘ + ↓ X 4

120 240 150 30 1

$$HP(k,k) = k!$$
$$HP(k,1) = 1$$

1

2 1

6 6 1

24 36 14 1

↘ + ↓ X 3

120 240 150 30 1

$$HP(k,k) = k!$$
$$HP(k,1) = 1$$

1

2 1

6 6 1

24 36 14 1

↘ + ↓ X 2

120 240 150 30 1

$$HP(k,k) = k!$$
$$HP(k,1) = 1$$

$$P_k(1)+P_k(2)+P_k(3)+\cdots+P_k(n)=P_{k+1}(n+1)$$

$$j^4 = 24P_4(j)+36P_3(j)+14P_2(j)+P_1(j)$$

$$P_k(1)+P_k(2)+P_k(3)+\cdots+P_k(n)=P_{k+1}(n+1)$$

$$j^4 = 24P_4(j)+36P_3(j)+14P_2(j)+P_1(j)$$

$$\begin{aligned}
1^4 + 2^4 + 3^4 + \cdots + n^4 &= \;?\\
24[P_4(1)+P_4(2)+P_4(3)+\cdots+P_4(n)] &= 24\ P_5(n+1)\\
36[P_3(1)+P_3(2)+P_3(3)+\cdots+P_3(n)] &= 36\ P_4(n+1)\\
14[P_2(1)+P_2(2)+P_2(3)+\cdots+P_2(n)] &= 14\ P_3(n+1)\\
[P_1(1)+P_1(2)+P_1(3)+\cdots+P_1(n)] &= P_2(n+1)
\end{aligned}$$

$$P_k(1)+P_k(2)+P_k(3)+\cdots+P_k(n)=P_{k+1}(n+1)$$

$$j^4 = 24P_4(j)+36P_3(j)+14P_2(j)+P_1(j)$$

$$\begin{aligned}
1^4 + 2^4 + 3^4 + \cdots + n^4 &= \;?\\
24[P_4(1)+P_4(2)+P_4(3)+\cdots+P_4(n)] &= 24\ P_5(n+1)\\
36[P_3(1)+P_3(2)+P_3(3)+\cdots+P_3(n)] &= 36\ P_4(n+1)\\
14[P_2(1)+P_2(2)+P_2(3)+\cdots+P_2(n)] &= 14\ P_3(n+1)\\
[P_1(1)+P_1(2)+P_1(3)+\cdots+P_1(n)] &= P_2(n+1)
\end{aligned}$$

$$1^4+2^4+3^4+\cdots+n^4$$
$$=24P_5(n+1)+36\,P_4(n+1)+14\,P_3(n+1)+P_2(n+1)$$

$$1^4 \quad + \quad 2^4 \quad + \quad 3^4 \quad +\cdots+ \quad n^4 \quad = \quad ?$$
$$24[P_4(1)+P_4(2)+P_4(3)+\cdots+P_4(n)]=24\ P_5(n+1)$$
$$36[P_3(1)+P_3(2)+P_3(3)+\cdots+P_3(n)]=36\ P_4(n+1)$$
$$14[P_2(1)+P_2(2)+P_2(3)+\cdots+P_2(n)]=14\ P_3(n+1)$$
$$[P_1(1)+P_1(2)+P_1(3)+\cdots+P_1(n)] \quad = \quad P_2(n+1)$$

$$1^4+2^4+3^4+\cdots+n^4$$
$$=24P_5(n+1)+36\,P_4(n+1)+14\,P_3(n+1)+P_2(n+1)$$
$$=\frac{1}{5}(n+1)n(n-1)(n-2)(n-3)$$
$$+\frac{3}{2}(n+1)n(n-1)(n-2)$$
$$+\frac{7}{3}(n+1)n(n-1)$$
$$+\frac{1}{2}(n+1)n$$

$$1^4+2^4+3^4+\cdots+n^4$$
$$=24P_5(n+1)+36\,P_4(n+1)+14\,P_3(n+1)+P_2(n+1)$$
$$=\frac{1}{5}(n+1)n(n-1)(n-2)(n-3) \qquad =\frac{n^5}{5}+\frac{n^4}{2}$$
$$+\frac{3}{2}(n+1)n(n-1)(n-2) \qquad +\frac{n^3}{3}-\frac{n}{30}$$
$$+\frac{7}{3}(n+1)n(n-1)$$
$$+\frac{1}{2}(n+1)n$$