Second Project for Math 377, fall, 2003

You get to pick your own project. Several possible topics are described below or you can come up with your own topic (subject to my approval). At most two people can jointly author the project report. At most two reports can be written on the same problem (first come, first served: Get the problem you want before two other people claim it). You may not jointly author with the same person as on the first project. **Progress report due Monday, November 10, 25% of 2nd project grade:** What have you tried? What has worked? What hasn't worked? What are you going to try next? **Final version due Friday, December 12, 75% of 2nd project grade.**

1. Wallis's formula (see Appendix A.1) states that

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots.$$

Give a complete proof of this identity, using exercises 2, 3, 4, 5, 7 to fill in the details. Find the product formula for

$$\int_0^1 (1 - x^p)^{1/q}\, dx$$

where p and q are arbitrary positive integers.

2. Stirling's formula (see Appendix A.4) states that

$$n! = n^n e^{-n} \sqrt{2\pi n}\, e^{E(n)}$$

where $\lim_{n\to\infty} E(n) = 0$. This error term, $E(n)$, can be approximated using the series

$$\frac{B_2}{1 \cdot 2n} + \frac{B_4}{3 \cdot 4n^3} + \frac{B_6}{5 \cdot 6n^5} + \cdots,$$

where B_{2k} is the $2k$th Bernoulli number (see Appendix A.2). Note that this is an alternating series.

Prove that $|B_{2m}| > 2 \cdot (2m)!/(2\pi)^{2m}$ (exercise 20 in Appendix A.3), and then use this fact to prove that the series diverges.

Although this series diverges, it can be used to get a very close approximation to $E(n)$. For any value of m, we have that

$$E(n) = \frac{B_2}{1 \cdot 2n} + \frac{B_4}{3 \cdot 4n^3} + \cdots + \frac{B_{2m}}{(2m-1)(2m)n^{2m-1}} + R(m,n),$$

where $R(m,n)$ lies strictly between 0 and $B_{2m+2}/(2m+1)(2m+2)n^{2m+1}$. (You don't have to prove this.)

If we take just the first two terms of this series, we see that

$$10! = 3628799.9714 * e^{R(2,10)}.$$

It follows that

$$3628799.9714 < 10! < 3628800.0103.$$

Since 10! is an integer, it must equal 3628800.

Is it always possible to use the divergent series of Bernoulli numbers to get an approximation to $n!$ that has an error of less than 0.5?

3. Stirling's formula is a special case of the Euler-Maclaurin formula. If f is an infinitely differentiable function, then

$$\sum_{i=1}^{n} f(i) = \frac{f(n)}{2} + \int_0^n f(x)\,dx + C + E(n)$$

where C is a constant and $E(n)$ is the error term that approaches 0 as n approaches infinity, and both C and $E(n)$ have specific asymptotic expansions in terms of the Bernoulli numbers.

Carefully state and prove the Euler-Maclaurin formula and give examples that show how it can be used.

4. There are many techniques for accelerating the convergence of an infinite series. One of these, based on Simpson's rule, is described in *The College Mathematics Journal*, volume 28, #5, November 1997. Explain the method, giving your own examples.

5. What can be said about the values of the derivative of a function when something is known about the function itself? It is a trivial task to find a function that takes arbitrarily large values with a derivative that remains bounded: consider, for example, $f(x) = x$. It is, however, more usual to expect that if the function becomes large, then so does its derivative: consider, for example, $f(x) = x^2$. One difference between the two examples is that the first is uniformly continuous, but the second is not. Is that the main difference?

If a function f is continuously differentiable and uniformly continuous, does it follow that the derivative f' is bounded?

6. A sequence $\{x_n\}$ of real numbers is said to be **Cesàro convergent** to x_0 if the sequence of its averages has the limit x_0; explicitly if

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x_0.$$

If we consider the summation

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots,$$

the sequence of partial sums is $\{1, 0, 1, 0, 1, 0, \ldots\}$. Show that this sequence is Cesàro convergent to $1/2$.

This is an important and justly famous concept. The first theorem of the subject is that ordinary convergence implies Cesàro convergence. That is: if $\lim_{n\to\infty} x_n = x_0$, then the sequence $\{x_n\}$ is Cesàro convergent to x_0. Prove this theorem.

What is an example of a sequence that is not Cesàro convergent? An easy one is given by $x_n = n$, but that may be unfair—it is too unbounded. What is an example of a bounded sequence that is not Cesàro convergent? One way to get one is to alternate longer and longer sequences of 0s and 1s: start with one 0, say, then put several 1s, then many 0s, then very many 1s, and so on. Replace "several", "many", "very many", etc. by precise values so that the sequence is not Cesàro convergent.

We can use the symbol

$$x_n \xrightarrow{C} x_0$$

to mean that $\{x_n\}$ is Cesàro convergent to x_0. We say that a function f is **Cesàro continuous** at $x = x_0$ if

$$x_n \xrightarrow{C} x_0 \qquad \text{implies} \qquad f(x_n) \xrightarrow{C} f(x_0).$$

Note that we have weakened the conclusion, but we have also weakened the hypothesis. Is every continuous function also Cesàro continuous? Is every Cesàro continuous function also continuous? Is $f(x) = x^2$ Cesàro continuous? Is it Cesàro continuous for any values of x?

7. What does

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}}$$

mean? How large is it? For what values of x does

$$x^{x^{x^{x^{\cdots}}}}$$

converge?

8. This is a problem posed by S. Ramanujan. Prove that the following expression converges and find its value:

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}.$$

Can you generalize what is going on here?

9. Let $\{a_n\}$ be a sequence whose subsequences $\{a_{2k}\}$, $\{a_{2k+1}\}$, and $\{a_{3k}\}$ are convergent.

   (a) Prove that the sequence $\{a_n\}$ is convergent.

   (b) Does the convergence of any two of these subsequences imply the convergence of the sequence $\{a_n\}$?

   (c) What can you say in general about when the convergence of certain subsequences implies (or does not necessarily imply) the convergence of the original sequences?

10. Prove that for every positive sequence $\{a_n\}$,

$$\limsup_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{a_n} \geq 4.$$

Find a sequence for which this lim sup equals 4.

11. Assume that $\sum_{n=1}^{\infty} a_n$ is a divergent series with positive summands. Let $\{S_n\}$ be the sequence of its partial sums.

   (a) Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{S_n} \quad \text{diverges}$$

   and

$$\sum_{n=1}^{\infty} \frac{a_n}{S_n^2} \quad \text{converges.}$$

   (b) Still assuming that $\sum_{n=1}^{\infty} a_n$ diverges, for what values of α does

$$\sum_{n=1}^{\infty} \frac{a_n}{S_n^{\alpha}} \quad \text{converge?}$$

   (c) If $\sum_{n=1}^{\infty} a_n$ converges, for what values of α does

$$\sum_{n=1}^{\infty} \frac{a_n}{S_n^{\alpha}} \quad \text{converge?}$$

12. Give an example of a bounded function on $[0, 1]$ which achieves neither its least upper bound nor its greatest lower bound on that inerval. Then give an example of a bounded function on $[0, 1]$ such that given any pair (a, b) with $0 \leq a < b \leq 1$, there is no $x \in [a, b]$ at which the function equals its greatest lower bound for the interval $[a, b]$.

13. A function $f : \mathbf{R} \to \mathbf{R}$ is *periodic* if there exists a positive number T (called the period), such that $f(x + T) = f(x)$ for all $x \in \mathbf{R}$.

(a) Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous periodic function with two *incommensurate* periods T_1 and T_2; that is, T_1/T_2 is irrational. Prove that f is a constant function. Give an example of a nonconstant periodic function with two incommensurate periods.

(b) Show that if $f : \mathbf{R} \to \mathbf{R}$ is nonconstant, periodic, and continuous, then it has a smallest positive period. This is called the *fundamental period* (e.g. $\sin x$ is periodic with period 4π, but its fundamental period is 2π.)

(c) Give an example of a nonconstant periodic function without a fundamental period.

(d) Prove that if $f : \mathbf{R} \to \mathbf{R}$ is a periodic function without a fundamental period, then the set of all periods of f is *dense* in $\mathbf{R}$. This means that given any $x \in \mathbf{R}$ and any $\epsilon > 0$, there is a period within ϵ of x.

(e) Prove that the result of part (b) remains true when the continuity of f on $\mathbf{R}$ is replaced by continuity at one point.

14. Suppose that $f : (0, \infty) \to (0, \infty)$ is uniformly continuous. Does this imply that

$$\lim_{x\to\infty} \frac{f(x + 1/x)}{f(x)} = 1?$$

15. A real function defined on an open set $A \subseteq \mathbf{R}$ is said to be *strongly differentiable* at $a \in A$ if

$$\lim_{\substack{(x_1,x_2)\to(a,a)\\ x_1\neq x_2}} \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f^*(a).$$

(a) Show that if f is strongly differentiable at a, then it is differentiable at a and $f^*(a) = f'(a)$. Show by example that the converse is not true.

(b) Prove that if f has a continuous derivative at a, then it is strongly differentiable at a.

(c) Does the strong differentiability of f at a imply the continuity of f' at this point?

16. We say that a sequence of functions $\{f_n\}$ defined on a set S *converges continuously* on S to the function f if for $x \in S$ and for every sequence $\{x_n\}$ of elements of S converging to x the sequence $\{f_n(x_n)\}$ converges to $f(x)$.

(a) Prove that if a sequence $\{f_n\}$ converges continuously on S to f, then for every sequence $\{x_n\}$ of elements of S converging to $x \in S$ and for every subsequence $\{f_{n_k}\}$,

$$\lim_{k\to\infty} f_{n_k}(x_k) = f(x).$$

(b) Prove that if $\{f_n\}$ converges continuously on S to f, then f is continuous on S (even if the f_n are not themselves continuous).

(c) Prove that if $\{f_n\}$ converges uniformly on S to the continuous function f, then $\{f_n\}$ converges continuously on S. Does the converse hold?

17. Let $f(x) = (x + (1/2)\sin(2x))$, $F(x) = (x + (1/2)\sin(2x))\, e^{\sin x}$. Both of these functions approach infinity as x approaches infinity. Their derivatives are $f'(x) = 1 + \cos(2x) = 2\cos^2 x$ and $F'(x) = 2\cos^2 x\ e^{\sin x} + (x + (1/2)\sin(2x))\ \cos x\ e^{\sin x}$. The ratio of these derivatives is

$$\frac{f'(x)}{F'(x)} = \frac{2\cos(x)}{2\cos(x)\ e^{\sin x} + (x + (1/2)\sin(2x))\, e^{\sin x}}.$$

As x approaches infinity, the numerator stays bounded between -2 and $+2$ while the denominator is always greater than or equal to

$$e^{-1}\left(-2 + (x - 1/2)\right),$$

which approaches infinity as x approaches infinity, and so

$$\lim_{x\to\infty} \frac{f'(x)}{F'(x)} = 0.$$

This is not equal to the limit of the ratio of $f(x)/F(x)$,

$$\lim_{x\to\infty} \frac{f(x)}{F(x)} = \lim_{x\to\infty} e^{-\sin x},$$

which oscillates between e^{-1} and e and so is undefined as x approaches infinity. In fact, there is a flaw in the proof of l'Hospital's rule in the infinite case. Where is that flaw and what additional assumption must be made so that we can apply l'Hospital's rule?