

Math 136, Fall '04
Project C: Shuffles

The purpose of this project is to determine the minimum number of perfect shuffles needed to return a deck of cards to its original order. First draft is due Friday, November 19. Final version is due Friday, December 10.

Read the enclosed article from *The Economist* of October 12, 1996 on “How to win at poker, and other science lessons.” This project is going to concentrate on a statement made near the top of the third column “that the number of outshuffles needed to get a pack back to its starting point is the smallest power of two—call it x —such that two raised to the power of x and then divided by a figure that is one less than the number of cards in the pack leaves a remainder of one.” In other words, if there are $2n$ cards in the deck, then the number of outshuffles needed to get a pack back to its starting point is the order of 2 modulo $2n - 1$.

1. What is the difference between an in-shuffle and an out-shuffle?
2. Prove that if you want to move the top card down by j positions in the deck, then this can be accomplished by reading the binary representation of j from left to right and doing an inshuffle at each 1 and an outshuffle at each 0. For example, to move a card down 13 places, the binary representation of 13 is 1101: do two inshuffles, an outshuffle, and then an inshuffle. You may find it helpful to think of the top card as being in position 0, the next card in position 1, and so on. You want to move the card from position 0 to position j . What happens to the card in position i on an inshuffle? What happens to the card in position i on an outshuffle?
3. Put the quoted sentence into mathematical notation. What does it say?
4. Prove the statement made in the quoted sentence. You may find it helpful to start with an example that uses a small deck of cards and then keep track of where each card goes. When you outshuffle a deck of $2n$ cards, what happens to the cards in each position? You'll need to consider two cases: when the position is 0 through $n - 1$ and when the position is n through $2n - 1$. Then find a way using modular arithmetic to give one rule that covers both cases.
5. Explain why the minimal number of outshuffles needed to restore a deck with $2n$ cards to its original order must be a divisor of $\phi(2n - 1)$. How many perfect outshuffles does it take to restore a deck of 104 cards to its original order? What about a deck with 148 cards?
6. Find and then prove a similar characterization for the minimal number of inshuffles needed to return a deck with $2n$ cards to its original order. For this problem, you want to relabel your positions so that the top card is in position 1. How many perfect inshuffles are needed to return a deck of 52 cards to its original order? What about a deck with 104 cards?