Math 136, Fall '04

Project A: The Stamp Problem

first draft due Friday, October 1; final version due Friday, October 22

Given positive integers a and b, this project looks at the question of what integers can and cannot be represented by $ma + nb$ where m and n are non-negative integers (integers greater than or equal to 0). It is sometimes referred to as the stamp problem because you can think of it as asking what postage amounts can you make if you are restricted to an unlimited supply of a¢ stamps and b¢ stamps.

1. In the list given below, cross out the postage amounts that *can* be made using 5¢ stamps and 8¢ stamps.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	29
30	31	32	33	34
35	36	37	38	39
⋮	⋮	⋮	⋮	⋮

2. What are the amounts that *cannot* be made? How do you know that there are no larger amounts that cannot be made?

3. If you start with 4¢ stamps and 9¢ stamps, what are the postage amounts that cannot be made? What if you had started with 4¢ stamps and 6¢ stamps?

4. When you have two kinds of stamps, when will there be a largest postage amount that you cannot make?

5. If you start with a¢ stamps and b¢ stamps, what is the largest postage amount that you cannot make?

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6. Justify your answer to the previous question. There are several pieces to this. Show that if a and b are relatively prime and you arrange the non-negative integers into a columns then

   (a) The smallest amount that can be made in any column must be a multiple of b.

   (b) Each column has at least one multiple of b in it.

   (c) The smallest multiple of b in each column is less than $a\,b$.

   (d) There must be a column for which the smallest multiple of b is $(a-1)b$.

   Once you have proven these statements, then use them to complete the justification of your formula.

7. If a and b are relatively prime, how many positive integers cannot be represented by $ma + nb$ where m and n are non-negative integers? Find a formula in terms of a and b and then prove that it is correct. **Hint:**

   (a) The **floor** of a rational number, x/y, is defined to be the largest integer less than or equal to x/y. It is written as $\lfloor x/y \rfloor$. For example, $\lfloor 12/5 \rfloor = 2$. Write the formula that you are looking for as a sum of floor functions.

   (b) Prove that

$$\left\lfloor \frac{b}{a} \right\rfloor + \left\lfloor \frac{2b}{a} \right\rfloor + \cdots + \left\lfloor \frac{(a-1)b}{a} \right\rfloor = \frac{b}{a} + \frac{2b}{a} + \cdots + \frac{(a-1)b}{a} - \frac{1}{a} - \frac{2}{a} - \cdots - \frac{a-1}{a}.$$

   (c) Now use this last equality to find a closed formula for the desired number of integers.