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Tensor Product Representations of General Linear Groups and Their Connections with Brauer Algebras

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For the complex general linear group $G = GL(r, \mathbb{C})$ we investigate the tensor product module $T = (\bigotimes^p V) \otimes (\bigotimes^q V^*)$ of p copies of its natural representation $V = \mathbb{C}^r$ and q copies of the dual space V^* of V. We describe the maximal vectors of T and from that obtain an explicit decomposition of T into its irreducible G-summands. Knowledge of the maximal vectors allows us to determine the centralizer algebra $\mathscr{C}$ of all transformations on T commuting with the action of G, to construct the irreducible $\mathscr{C}$-representations, and to identify $\mathscr{C}$ with a certain subalgebra $\mathscr{B}^{(r)}_{p,q}$ of the Brauer algebra $\mathscr{B}^{(r)}_{p+q}$. © 1994 Academic Press, Inc.

INTRODUCTION

Let G denote the general linear group $GL(r, \mathbb{C})$ of $r \times r$ invertible complex matrices and let V be the space $\mathbb{C}^r$ of $r \times 1$ complex matrices. Then G acts naturally on V by matrix multiplication making V into a G-module. The tensor space $T = \bigotimes^p V$ inherits a G-module structure as does V^*, the dual space of V. The symmetric group $S_{\mathscr{P}}$ on $\mathscr{P} = \{1, \dots, p\}$ acts on T by permuting the factors. When $r \geqslant p$, the group algebra $\mathbb{C}[S_{\mathscr{P}}]$ is the centralizer algebra $\mathrm{End}_G(T)$ of all transformations on T commuting with the action of G, and the associative envelope of G on T is the centralizer algebra $\mathrm{End}_{S_{\mathscr{P}}}(T)$. That fundamental result, commonly referred to as Schur–Weyl duality, links in a critical way the representation theories of the general linear and symmetric groups. The projection maps of T onto its irreducible G-summands are primitive idempotents in $\mathbb{C}[S_{\mathscr{P}}]$, which are described in the classical work of Schur [Sc1, Sc2] and Weyl [Wey]. These idempotents, or Young symmetrizers as they are often called,

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enabled Weyl to give an explicit decomposition of T into irreducible summands and to study irreducible representations and invariants of G.

In this paper we investigate the tensor product $T=(\otimes^p V)\otimes(\otimes^q V^*)$ of p copies of V and q copies of the dual space V^*. Our starting point is a result obtained by Stembridge [St1, St2] using insertion schemes, which determines the highest weights of the irreducible G-summands of T. In Section 1 we convert Stembridge's highest weight labels to pairs $(\tau, (\underline{m}, \underline{n}))$ consisting of a set $(\underline{m}, \underline{n}) = \{(m_1, n_1), ..., (m_k, n_k) \mid 1 \leqslant m_i \leqslant p,\ p+1 \leqslant n_i \leqslant p+q\}$ and a standard rational tableau τ having entries in $\underline{m}^c \cup \underline{n}^c \subseteq \{1, ..., p+q\}$. This allows us to identify the maximal vectors of T relative to the Lie algebra $\mathscr{G} = gl(r, \mathbb{C})$ of G and to explicitly decompose T into irreducible $\mathscr{G}$-summands (hence G-summands) by methods similar to those in [BBL]. In particular, the group $S_{\mathscr{P}} \times S_{\mathscr{Q}}$, where $\mathscr{Q} = \{p+1, ..., p+q\}$ acts on T as do certain transformations $c_{m,n}$ termed contraction maps which are defined in Section 2. The algebra $\mathscr{A}$ of endomorphisms of T generated by $S_{\mathscr{P}} \times S_{\mathscr{Q}}$ and the contraction maps lies in the centralizer algebra $\mathscr{C} = \mathrm{End}_{\mathscr{G}}(T)$ (also in $\mathrm{End}_G(T)$). If $v_1, ..., v_r$ is the standard basis for V, and $v_1^*, ..., v_r^*$ is the dual basis in V^*, then we establish the following:

THEOREMS 2.7, 2.11. *Let $(\tau, (\underline{m}, \underline{n}))$ consist of a set $(\underline{m}, \underline{n}) = \{(m_1, n_1), ..., (m_k, n_k) \mid 1 \leqslant m_i \leqslant p,\ p+1 \leqslant n_i \leqslant p+q\}$ and a standard rational tableau $\tau = (\tau^+, \tau^-)$, where τ^+ is a standard tableau with entries in $\underline{m}^c = \mathscr{P} - \underline{m}$ and τ^- is a standard tableau with entries in $\underline{n}^c = \mathscr{Q} - \underline{n}$. Assume*

$$\beta_{\tau,\underline{m},\underline{n}} = u_1 \otimes \cdots \otimes u_p \otimes u_{p+1}^* \otimes \cdots \otimes u_{p+q}^*$$

is the simple tensor whose factors are defined by

$$u_l = \begin{cases} v_1 & \text{if } l \in \underline{m} \\ v_j & \text{if } l \in \underline{m}^c \text{ and } l \text{ in the } j\text{th row of } \tau^+, \end{cases}$$

$$u_l^* = \begin{cases} v_1^* & \text{if } l \in \underline{n} \\ v_{r-j+1}^* & \text{if } l \in \underline{n}^c \text{ and } l \text{ in the } j\text{th row of } \tau^-. \end{cases}$$

Then $t_{\tau,\underline{m},\underline{n}} \stackrel{\text{def}}{=} y_\tau c_{\underline{m},\underline{n}} \beta_{\tau,\underline{m},\underline{n}}$, where y_τ is the product $y_{\tau^+} y_{\tau^-}$ of the associated Young symmetrizers and $c_{\underline{m},\underline{n}}$ is the product $c_{m_1,n_1} \cdots c_{m_k,n_k}$ of the contraction maps, is a maximal vector. The vectors $t_{\tau,\underline{m},\underline{n}}$ as k ranges over all possible values $0, ..., \min(p, q)$ determine a basis for the space of all maximal vectors of T when $r \geqslant p+q$.

As a result, T has the following decomposition:

THEOREM 2.12. *If $r \geqslant p+q$, then*

$$T \cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} \frac{p!\, q!}{k!\, h(\mu)\, h(\nu)} V([\mu, \nu]_r),$$

where $V([\mu, \nu)_r)$ denotes the finite dimensional irreducible $\mathcal{G}$-module having the r-staircase $[\mu, \nu]_r$ as its highest weight; μ, ν are partitions of $p-k$, $q-k$, respectively; and $h(\mu)$, $h(\nu)$ are the corresponding hook lengths.

It is apparent from this theorem that the decomposition is the "same" for all $r \geqslant p+q$ provided the highest weights of the irreducible summands are interpreted appropriately. Our description of the maximal vectors and the decomposition specializes to the classical results [Sc1, Sc2, Wey, BBL] in the $q=0$ case. When $p=q$, the tensor space T is isomorphic to the p-fold tensor product $\bigotimes^p \mathcal{G}$ of the adjoint representation of $\mathcal{G}$, as $V \otimes V^* \cong \mathcal{G}$. Hanlon [Han1, Han2] uses the theory of Specht modules for the symmetric group $S_{\not p}$ to construct the maximal vectors in $\bigotimes^p \mathcal{G}$. Our results can be shown to reduce to Hanlon's in this particular situation. As Hanlon proves in [Han3], the stable behavior of the decomposition of $T = \bigotimes^p \mathcal{G}$ as $r \to \infty$ can be applied to the problem of describing the $GL(r, \mathbb{C})$-module structure of the Lie algebra cohomology of $gl(r, \mathfrak{a})$ with trivial coefficients for $\mathfrak{a}$ a graded associative $\mathbb{C}$-algebra.

The maximal vectors in T of a given weight constitute a module for the centralizer algebra $\mathcal{C}$, hence also an $\mathcal{A}$-module. In Section 3 we give a specific description of the action of $\mathcal{A}$ on the maximal vectors. This allows us to prove in Section 4 that the space of all maximal vectors in T of a given weight is an irreducible $\mathcal{A}$-module. We show that the modules corresponding to different weights are nonisomorphic. Those results enable us in Theorem 5.3 and Corollary 5.4 to conclude when $r \geqslant p+q$ that (i) $\mathcal{A} = \mathcal{C} = \mathrm{End}_{\mathcal{G}}(T)$ (Compare [Koi, Theorem 1.1].) and (ii) the modules of maximal vectors form a complete set of representatives for the isomorphism classes of irreducible $\mathcal{A}$-modules. From that information it is easy to determine the precise dimension of the centralizer algebra when $r \geqslant p+q$.

Brauer [Bra] first introduced what are now called the Brauer algebras (and what we denote $\mathcal{B}_f^{(x)}$ for f an integer and x an indeterminant) to study representations and invariants of the orthogonal and symplectic groups. When x is specialized to the integer r, there is a representation $\mathcal{B}_f^{(r)} \to \mathrm{End}_{O(r,\mathbb{C})}(\bigotimes^f \mathbb{C}^r)$ onto the centralizer algebra of the orthogonal group $O(r, \mathbb{C})$ acting on the tensor space $\bigotimes^f \mathbb{C}^r$. Therefore in the orthogonal case, the Brauer algebra plays the role of the group algebra of the symmetric group in Schur–Weyl duality. We identify the centralizer algebra $\mathcal{C}$ of $\mathcal{G}$ on $T = (\bigotimes^p V) \otimes (\bigotimes^q V^*)$ with a certain subalgebra $\mathcal{B}_{p,q}^{(r)}$ of the Brauer algebra $\mathcal{B}_{p+q}^{(r)}$, and prove:

THEOREM 5.9. *If $r \geqslant p+q$, then as a bimodule for $\mathcal{B}_{p,q}^{(r)} \times \mathcal{G}$,*

$$T \cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} M_{\mu,\nu} \otimes V([\mu, \nu]_r),$$

where $M_{\mu,\nu}$ is the irreducible $\mathscr{B}_{p,q}^{(r)}$-module labeled by the pair of partitions μ, ν and $V([\mu,\nu]_r)$ is as in Theorem 2.12.

1. Standard Rational Tableaux and Two Line Arrays

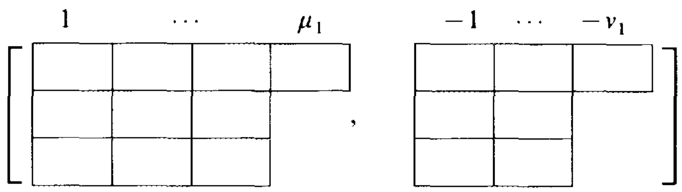
Let V be complex r-space $\mathbb{C}^r$ viewed as $r\times 1$ matrices, and let $v_1, v_2, ..., v_r$ denote the canonical basis of V. The general linear group $G=GL(r,\mathbb{C})$ of all $r\times r$ complex invertible matrices acts naturally on V by matrix multiplication making V into a G-module. The dual space V^* of V inherits a G-module structure given by $(g\cdot u^*)v=u^*(g^{-1}\cdot v)$. Let $v_1^*, v_2^*, ..., v_r^*$ denote the dual basis to $v_1, v_2, ..., v_r$ in V^*. We identify v_i^* with the $1\times r$ matrix having 1 in its ith column and 0 everywhere else. We fix integers $p, q\geqslant 0$ such that $p+q>0$ and choose subsets $\mathscr{P}, \mathscr{Q}\subseteq\{1,2,...,p+q\}$ with $|\mathscr{P}|=p$, $|\mathscr{Q}|=q$, and $\mathscr{P}\cup\mathscr{Q}=\{1,2,...,p+q\}$. Then the tensor product $T=T^1\otimes\cdots\otimes T^{p+q}$, where $T^i=V$ for $i\in\mathscr{P}$ and $T^i=V^*$ for $i\in\mathscr{Q}$ is also a G-module where the action is afforded by $g\cdot(t_1\otimes t_2\otimes\cdots\otimes t_{p+q})=g\cdot t_1\otimes g\cdot t_2\otimes\cdots\otimes g\cdot t_{p+q}$ for $g\in G$ and $t_i\in T^i$. The module T is completely reducible and its irreducible summands are necessarily rational G-representations (see [Sc1, Sc2]). Let H be the Cartan subgroup of diagonal matrices in G and let ε_i denote the map which takes a matrix to its (i,i) entry. An irreducible rational G-module can be indexed by its highest weight relative to H, which is an integral linear combination $\lambda=\lambda_1\varepsilon_1+\cdots+\lambda_r\varepsilon_r$ whose coefficients satisfy $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_r$. We identify λ with the sequence $(\lambda_1,\lambda_2,...,\lambda_r)\in\mathbb{Z}^r$ of its coefficients, and following Stembridge [St2] we say that λ is a *staircase of height* r or simply an *r-staircase*. The λ_i are the *parts* of λ. We adopt the notation $V(\lambda)$ for the irreducible rational G-module corresponding to the r-staircase λ. For convenience we collect like parts of an r-staircase. Thus if $r=7$, then $(4,3^2,0,(-2)^2,-3)$ is our shorthand for $(4,3,3,0,-2,-2,-3)$.

An r-staircase μ whose parts are nonnegative determines a *partition* $\{\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_r\geqslant 0\}$. The *length* $l(\mu)$ of the partition μ is the number of nonzero parts. If $|\mu|=^{\text{def}}\mu_1+\cdots+\mu_{l(\mu)}=N$, then μ is a partition of N, which we signify by writing $\mu\vdash N$. We use $\Pi(N)$ to denote the set of all partitions of N. Corresponding to μ is its *Young frame* or *Ferrers diagram* $\mathscr{F}(\mu)$ having N boxes with μ_i boxes in the ith row for $i=1,...,l(\mu)$.

Associated to the r-staircase $\lambda=(\lambda_1,\lambda_2,...,\lambda_r)$, where $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_i>0$ and $0>\lambda_j\geqslant\lambda_{j+1}\geqslant\cdots\geqslant\lambda_r$, are the two partitions, $\mu=\{\lambda_1\geqslant\cdots\geqslant\lambda_i>0\}$ and $\nu=\{-\lambda_r\geqslant\cdots\geqslant-\lambda_j>0\}$, termed the *positive and negative parts* of λ. Conversely, any pair of partitions $\mu=\{\mu_1\geqslant\cdots\geqslant\mu_{l(\mu)}>0\}$ and $\nu=\{\nu_1\geqslant\cdots\geqslant\nu_{l(\nu)}>0\}$ determines an r-staircase $\lambda=(\mu_1,...,\mu_{l(\mu)},0,...,0,-\nu_{l(\nu)},...,-\nu_1)\in\mathbb{Z}^r$ for each $r\geqslant l(\mu)+l(\nu)$, where the parts have been separated by $r-l(\mu)-l(\nu)$ zeros. Thus, in what follows we use the terms

r-staircase and pair of partitions interchangeably, however, always with the implicit understanding that r is sufficiently large. We adopt the notation $\lambda = [\mu, \nu]_r$ to mean that the r-staircase λ is obtained from the pair of partitions $[\mu, \nu]$ by stretching them by adding $r - l(\mu) - l(\nu)$ zeros, and we write either $V(\lambda)$ or $V([\mu, \nu]_r)$ for the corresponding irreducible representation. We use $[\mu, \nu] \in \Pi(N, N')$ to abbreviate $\mu \in \Pi(N)$ and $\nu \in \Pi(N')$.

The frame $\mathscr{F}(\lambda)$ of $\lambda = [\mu, \nu]_r$ is simply the pair of frames $[\mathscr{F}(\mu), \mathscr{F}(\nu)]$, and $l(\lambda) =^{\text{def}} l(\mu) + l(\nu)$. Our convention is that the rows of the r-staircase λ are labeled by numbering the rows of μ with $1, 2, ..., l(\mu)$ from top to bottom, and the rows of ν with $1, ..., l(\nu)$ from top to bottom. The columns of λ are indexed by first numbering the columns of μ by $1, 2, ..., \mu_1$ from left to right, and then the columns of ν by $-1, ..., -\nu_1$ from left to right. Therefore the diagram of $\lambda = (4, 3^2, 0, (-2)^2, -3)$ is

$\mathscr{F}(\lambda) =$ 

Suppose that λ contains a box at position (i, j). If $j > 0$, and λ has no box at position $(i, j+1)$ or $(i+1, j)$, then (i, j) is a *corner* of λ. Similarly, if $j < 0$ and λ has no box at $(i, j-1)$ or $(i+1, j)$, then (i, j) is a corner. In the above example $(1, 4)$, $(3, 3)$, $(1, -3)$, $(3, -2)$ are corners.

The G-modules which occur when T is decomposed into irreducible summands, or more precisely the r-staircases which label them, have been described by Stembridge in [St1, St2]. Since this is the starting point for our studies of T, we briefly review Stembridge's result.

DEFINITION 1.1. Suppose λ, λ' are r-staircases. If $\lambda_i \geqslant \lambda'_i$ for each i, then we say λ *contains* λ' and write $\lambda \supseteq \lambda'$. When $\lambda \supseteq \lambda'$, then $|\lambda/\lambda'| =^{\text{def}} \sum_{i=1}^{r} \lambda_i - \lambda'_i$.

DEFINITION 1.2. Let $\Lambda = \lambda^0, \lambda^1, ..., \lambda^{p+q}$ be a sequence of r-staircases with $\lambda^0 = (0, 0, ..., 0)$. Then Λ is a *$(\mathscr{P}, \mathscr{Q})$ up-down r-staircase tableau* if

(1) $\lambda^i \supset \lambda^{i-1}$ and $|\lambda^i/\lambda^{i-1}| = 1$ for each $i \in \mathscr{P}$, and

(2) $\lambda^i \subset \lambda^{i-1}$ and $|\lambda^{i-1}/\lambda^i| = 1$ for each $i \in \mathscr{Q}$.

The *shape* of Λ is λ^{p+q}.

THEOREM 1.3 [St2]. *The irreducible G-summands of T are in one-to-one correspondence with the $(\mathscr{P}, \mathscr{Q})$ up-down r-staircase tableaux Λ. The*

481/166/3-8

summand corresponding to A is isomorphic to $V(\lambda^{p+q})$, where λ^{p+q} is the terminal r-staircase (shape) of A.

We translate this theorem into a different form that gives more explicit information about the summands. To accomplish this we need the notion of a rational tableau.

DEFINITION 1.4. Suppose $[\mu, \nu]$ is a pair of partitions, and let $\mathscr{K}$ and $\mathscr{L}$ be two disjoint sets of positive integers. A $(\mathscr{K}, \mathscr{L})$ *rational tableau* τ *of shape* $[\mu, \nu]$ is a filling-in of the boxes of μ with integers from $\mathscr{K}$ and the boxes of ν with integers from $\mathscr{L}$. The tableau τ is said to be *standard* if the numbers in μ and ν strictly increase from left to right across each row and strictly increase top to bottom down in each column. The set of numbers in τ is $\mathcal{N}(\tau)$.

For example, suppose that $\mu = \{4, 3^2\}$ and $\nu = \{3, 2^2\}$ are the two partitions, and $\mathscr{K} = \{2, 4, 6, 8, 10, 12, 16, 18, 19, 20\}$ and $\mathscr{L} = \{1, 3, 5, 7, 9, 11, 13, 15\}$ the sets of integers. Then the following is a $(\mathscr{K}, \mathscr{L})$ standard rational tableau τ of shape $[\mu, \nu]$:

$$\tau = \left[\begin{array}{|c|c|c|c|} \hline 2 & 4 & 8 & 12 \\ \hline 6 & 10 & 14 \\ \cline{1-3} 18 & 19 & 20 \\ \cline{1-3} \end{array}\ ,\ \begin{array}{|c|c|c|} \hline 1 & 5 & 7 \\ \hline 3 & 11 \\ \cline{1-2} 9 & 15 \\ \cline{1-2} \end{array} \right].$$

We let τ^+ and τ^- denote the μ and ν portions of the rational tableau τ and write $\tau \in \mathscr{T}(\mathscr{K}, \mathscr{L})$. When we want to indicate that the underlying frame of τ is the pair $[\mu, \nu]$, we write $\tau \in \mathscr{T}_{\mu,\nu}(\mathscr{K}, \mathscr{L})$. Similarly, we use $\tau \in \mathscr{S}\mathscr{T}(\mathscr{K}, \mathscr{L})$ and $\tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\mathscr{K}, \mathscr{L})$ for τ a standard rational tableau. Rows and columns of a rational tableau are numbered according to our convention above.

Now, we describe an algorithm that translates an up-down r-staircase tableau into a pair consisting of standard rational tableau and a two line array. A similar translation scheme has been constructed by Sundaram [Su1, Su2] for dealing with tensor products of the natural representation of the orthogonal and symplectic groups.

DEFINITION 1.5. Let $\mathscr{P}$ and $\mathscr{Q}$ denote disjoint sets with $\mathscr{P} \cup \mathscr{Q} = \{1, ..., p+q\}$. Suppose τ is a $(\mathscr{P}, \mathscr{Q})$ standard rational tableau of shape $[\mu, \nu]$, and let (i, j) be a corner in τ. *Elimination* of (i, j) from τ produces an integer $x \in \{1, 2, ..., p+q\}$ and a $(\mathscr{P}, \mathscr{Q})$ standard tableau τ' of shape $[\mu', \nu']$, where either $|\mu/\mu'| = 1$ and $\nu' = \nu$ or $|\nu/\nu'| = 1$ and $\mu' = \mu$. In the case that $j > 0$ the procedure consists of the following steps:

(1) Delete the box at (i, j) from τ, and let x_j be the integer it contained.

(2) If x_{j-1} is the largest integer in column $j-1$ such that $x_{j-1} < x_j$, then insert x_j into column $j-1$ by displacing x_{j-1}.

(3) Insert x_{j-1} into column $j-2$ by the same rule, displacing x_{j-2}.

(4) Continue this process until an integer x_1 is displaced from column 1. Define x to be x_1 and τ' to be the final tableau.

When $j < 0$, elimination is defined analogously, but this time producing a sequence $x_j, x_{j+1}, ..., x_{-1}$ and assigning x the value x_{-1}.

Insertion, which we introduce next, is a natural adaptation of Schensted column-insertion, and elimination and insertion are easily seen to be inverses of one another.

DEFINITION 1.6. Let τ' be a $(\mathscr{P}, \mathscr{Q})$ standard rational tableau of shape $[\mu', \nu']$ and let x be an element of $\{1, 2, ..., p+q\}$ distinct from the entries of τ'. *Insertion of x into τ'* produces a $(\mathscr{P}, \mathscr{Q})$ standard rational tableau τ of shape $[\mu, \nu]$, where either $|\mu/\mu'| = 1$ and $\nu = \nu'$ or $|\nu/\nu'| = 1$ and $\mu = \mu'$. When $x \in \mathscr{P}$, insertion is defined as follows:

(1) If x_1 is the smallest integer in column 1 of τ'^+ such that $x_1 > x$, then insert x into column 1 by displacing x_1; if there is no such x_1, adjoin x to the bottom of column 1.

(2) Insert x_1 into column 2 of τ'^+ according to the same rule, displacing x_2.

(3) Continue in this way until an integer is adjoined to the bottom of a (possibly empty) column. Define τ to be the $(\mathscr{P}, \mathscr{Q})$ standard rational tableau whose left tableau is the resulting modification of τ'^+ and whose right tableau is τ'^-.

When $x \in \mathscr{Q}$ we define insertion analogously, but with τ'^- in place of τ'^+ and columns $-1, -2, ...$ in place of $1, 2, ...$, respectively.

DEFINITION 1.7. Let

$$L = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}$$

be a $2 \times k$ matrix. Say that L is a $(\mathscr{P}, \mathscr{Q})$ *two line array* provided

(1) $a_1, a_2, ..., a_k, b_1, b_2, ..., b_k$ are distinct elements of $\mathscr{P} \cup \mathscr{Q} = \{1, 2, ..., p+q\}$,

(2) $a_1 < a_2 < \cdots < a_k$,

(3) $a_i > b_i$ for each i, and

(4) $a_i \in \mathscr{P}$ if and only if $b_i \in \mathscr{Q}$, and vice-versa.

Write $\mathscr{N}(L)$ for the set of numbers $\{a_1, a_2, ..., a_k, b_1, b_2, ..., b_k\}$.

EXAMPLE 1.8. If $\mathscr{P} = \{1, 3, 5, 7\}$ and $\mathscr{Q} = \{2, 4, 6, 8, 10\}$, then $L = \binom{2\ 5\ 7}{1\ 4\ 6}$ is a $(\mathscr{P}, \mathscr{Q})$ two line array.

Now we have the necessary tools to reformulate Theorem 1.3.

Suppose $\Lambda = \lambda^0, \lambda^1, ..., \lambda^{p+q}$ is a $(\mathscr{P}, \mathscr{Q})$ up-down r-staircase tableau of shape $\lambda^{p+q} = [\mu, \nu]_r$. The following algorithm $\mathfrak{A}$ operates on Λ to produce a $(\mathscr{P}, \mathscr{Q})$ standard rational tableau τ of shape $[\mu, \nu]$ and a $(\mathscr{P}, \mathscr{Q})$ two line array L:

(1) Initially set τ to be the empty $(\mathscr{P}, \mathscr{Q})$ standard rational tableau of shape $[\emptyset, \emptyset]$ and L to be the empty $(\mathscr{P}, \mathscr{Q})$ two line array.

(2) Assume $\lambda^1, \lambda^2, ..., \lambda^{i-1}$ have been processed, yielding corresponding values of τ and L. If λ^i is constructed from λ^{i-1} by adding a box, adjoin a box containing i to τ at the same position. Otherwise λ^i is obtained from λ^{i-1} by deleting a corner. In this case, use elimination to remove the corresponding corner from τ. If x is the number produced by the elimination, add i and x to the right end of L with i in the first row and x in the second. In either case the shape of τ is λ^i.

(3) The final values of τ and L are the result of the procedure.

EXAMPLE 1.9. Let $\mathscr{P} = \{1, 3, 5\}$ and $\mathscr{Q} = \{2, 4\}$. Displayed below are a sample Λ and the corresponding values for τ and L.

	λ^0	λ^1	λ^2	λ^3	λ^4	λ^5
Λ:	$[\emptyset, \emptyset]_r$,	$[\{1\}, \emptyset]_r$,	$[\{1\}, \{1\}]_r$	$[\{1\}, \emptyset]_r$,	$[\emptyset, \emptyset]_r$,	$[\{1\}, \emptyset]_r$
τ:	$[\emptyset, \emptyset]$,	$\left[\boxed{1}, \emptyset\right]$,	$\left[\boxed{1}, \boxed{2}\right]$	$\left[\boxed{1}, \emptyset\right]$,	$[\emptyset, \emptyset]$,	$\left[\boxed{5}, \emptyset\right]$
L:	$(\ )$	$(\ )$	$(\ )$	$\binom{3}{2}$	$\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

Remark. It is easy to see that $\mathscr{N}(\tau) \cap \mathscr{N}(L) = \emptyset$ and $\mathscr{N}(\tau) \cup \mathscr{N}(L) = \{1, 2, ..., p+q\}$ for the τ and L produced by algorithm $\mathfrak{A}$. We can use insertion to define an inverse procedure $\mathfrak{B}$. Algorithm $\mathfrak{B}$ consists of $p+q$ steps which are numbered from $p+q$ to 1 and which produce the r-staircases $\lambda^{p+q}, \lambda^{p+q-1}, ..., \lambda^1$ successively. It begins by setting $\lambda^{p+q} = [\mu, \nu]_r$, where $[\mu, \nu]$ is the shape of τ. If at the ith step the number i appears in τ, then we remove the box containing it, and the shape of the resulting tableau gives the corresponding r-staircase λ^{i-1}. If instead i appears in the top row of L, then the number in the bottom row is the result of an elimination. We insert that number into the tableau, and the shape of the result of the insertion is the r-staircase λ^{i-1}. For example, suppose that $\mathscr{P}=\{2,4,7,8\}$ and $\mathscr{Q}=\{1,3,5,6\}$, and assume

$$\tau=\left[\begin{array}{|c|}\hline 2\\ \hline 7\\ \hline\end{array}\,,\ \begin{array}{|c|c|}\hline 3 & 6\\ \hline\end{array}\right] \qquad \text{and} \qquad L=\begin{pmatrix}5 & 8\\ 4 & 1\end{pmatrix}.$$

Then

$$\Lambda=\overbrace{[\emptyset,\emptyset]_r}^{\lambda^0},\ \overbrace{[\emptyset,\{1\}]_r}^{\lambda^1},\ \overbrace{[\{1\},\{1\}]_r}^{\lambda^2},\ \overbrace{[\{1\},\{2\}]_r}^{\lambda^3},\ \overbrace{[\{1^2\},\{2\}]_r}^{\lambda^4},$$

$$\overbrace{[\{1\},\{2\}]_r}^{\lambda^5},\ \overbrace{[\{1\},\{3\}]_r}^{\lambda^6},\ \overbrace{[\{1^2\},\{3\}]_r}^{\lambda^7},\ \overbrace{[\{1^2\},\{2\}]_r}^{\lambda^8}.$$

Procedure $\mathfrak{B}$ may produce at some step a staircase λ^i for which $l(\lambda^i)>r$. However, if τ and L are the result of $\mathfrak{A}$, this will not happen. If we assume that $r\geqslant p+q$, again there is no problem with the lengths, and in that case algorithm $\mathfrak{A}$ determines a bijection between the $(\mathscr{P},\mathscr{Q})$ up-down r-staircase tableaux and the pairs (τ, L) having the following characteristics:

(1.10) (1) τ is a $(\mathscr{P},\mathscr{Q})$ standard rational tableau of shape $[\mu,\nu]$,

(2) L is a $(\mathscr{P},\mathscr{Q})$ two line array,

(3) $\mathcal{N}(\tau)\cap\mathcal{N}(L)=\emptyset$, and

(4) $\mathcal{N}(\tau)\cup\mathcal{N}(L)=\{1,2,\ldots,p+q\}$.

Thus, algorithm $\mathfrak{A}$ enables us to label the irreducible summands of T with the pairs (τ, L) satisfying (1.10). However, if a pair (τ, L) gives a staircase λ^i with $l(\lambda^i)>r$ at some step i when the inverse algorithm $\mathfrak{B}$ is applied to (τ, L), then the irreducible summand corresponding to (τ, L) does not occur in the decomposition. To summarize we have:

THEOREM 1.11. *The irreducible G-summands of T are in one-to-one correspondence with the pairs (τ, L) which satisfy* (1.10) *and which correspond to $(\mathscr{P},\mathscr{Q})$ up-down r-staircase tableaux. The summand corresponding to (τ, L) is isomorphic to $V([\mu,\nu]_r)$, where $[\mu,\nu]$ is the shape of τ. In particular when $r\geqslant p+q$, the irreducible G-summands of T are in one-to-one correspondence with the pairs (τ, L) satisfying* (1.10).

2. THE DECOMPOSITION OF $(\otimes^p V)\otimes(\otimes^q V^*)$

In this section we use the result of Theorem 1.11 to gain explicit information about the tensor product $T = T^1 \otimes T^2 \otimes \cdots \otimes T^{p+q}$, where each T^i equals V or V^*, and there are p copies of V and q of V^*. Because the tensor products with p copies of V and q copies of V^* are all isomorphic, there is no loss in generality in assuming that $\mathscr{P} = \{1, 2, ..., p\}$ and $\mathscr{Q} = \{p+1, p+2, ..., p+q\}$, and so for ease in notation, we do so throughout the remainder of the work. Thus, we write $T = (\otimes^p V)\otimes(\otimes^q V^*)$. Every finite dimensional module for $G = GL(r, \mathbb{C})$ is also a module for its Lie algebra $\mathscr{G} = gl(r, \mathbb{C})$, and irreducible G-modules are irreducible $\mathscr{G}$-modules and vice versa. (See, for example, [M].) It is easier in certain arguments to adopt the point of view that we are working with modules for the Lie algebra $\mathscr{G}$, and so henceforth we will regard T as a $\mathscr{G}$-module. To split T into irreducible $\mathscr{G}$-modules we define two kinds of transformations on T: symmetrizers which involve permutations of the factors in $\otimes^p V$ and in $\otimes^q V^*$ and contractions which use the dual pairing between V and V^*.

Let $S_{\mathscr{P}}$ be the group of permutations on $\mathscr{P} = \{1, ..., p\}$ and $S_{\mathscr{Q}}$ the group of permutations on $\mathscr{Q} = \{p+1, ..., p+q\}$. Then $S_{\mathscr{P}} \times S_{\mathscr{Q}} \subseteq S_{\mathscr{P}\cup\mathscr{Q}}$, where $S_{\mathscr{P}\cup\mathscr{Q}}$ is the group of permutations on $\mathscr{P}\cup\mathscr{Q} = \{1, 2, ..., p+q\}$. For $\sigma \in S_{\mathscr{P}} \times S_{\mathscr{Q}}$ define

$$\begin{aligned}\sigma(u_1 \otimes \cdots \otimes u_p \otimes u^*_{p+1} \otimes \cdots \otimes u^*_{p+q})\\ = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(p)} \otimes u^*_{\sigma^{-1}(p+1)} \otimes \cdots \otimes u^*_{\sigma^{-1}(p+q)},\end{aligned}$$

where $u_i \in V$ for $i = 1, ..., p$ and $u^*_j \in V^*$ for $j = p+1, ..., p+q$. Note that σ never interchanges factors in $\otimes^p V$ with factors in $\otimes^q V^*$. Moreover, the action of $S_{\mathscr{P}} \times S_{\mathscr{Q}}$ commutes with the action of $\mathscr{G}$ (and also G) on T.

For $1 \leqslant m \leqslant p$ and $p+1 \leqslant n \leqslant p+q$ the *contraction map* $c_{m,n}$ is the linear transformation on T defined by

$$\begin{aligned}c_{m,n}(u_1 \otimes \cdots \otimes u_p \otimes u^*_{p+1} \otimes \cdots \otimes u^*_{p+q})\\ = u^*_n(u_m) \sum_{j=1}^{r} u_1 \otimes \cdots \otimes u_{m-1} \otimes v_j \otimes \cdots \otimes u_p \otimes u^*_{p+1} \otimes \cdots \otimes u^*_{n-1}\\ \otimes v^*_j \otimes \cdots \otimes u^*_{p+q}.\end{aligned}$$

(where the v_j, v^*_j are elements of the canonical bases.) Using the fact that $\sum_{j=1}^{r} v_j \otimes v^*_j$ spans a trivial module for $\mathscr{G}$ and G (compare [BBL, p. 19]), it is easy to check that $c_{m,n}$ commutes with the actions of both $\mathscr{G}$ and G, and $c_{m,n}((\otimes^p V)\otimes(\otimes^q V^*)) \cong (\otimes^{p-1} V)\otimes(\otimes^{q-1} V^*)$.

The following useful properties are readily verified. (See [BBL, Lemma 2.18].)

LEMMA 2.1. (a) $\sigma c_{m,n} = c_{\sigma(m),\sigma(n)}\sigma$ *for all* m, n *and all* $\sigma \in S_{\mathscr{P}} \times S_{\mathscr{Q}}$.

(b) $c_{k,l}c_{m,n} = c_{m,n}c_{k,l}$ *for all distinct* k, l, m, n.

(c) $c_{m,n}^2 = rc_{m,n}$.

(d) $c_{l,m}c_{l,n} = c_{l,m}(m\ \ n) = (m\ \ n)c_{l,n}$ *for all distinct* l, m, n.

(e) $c_{k,m}c_{l,m} = c_{k,m}(k\ \ l) = (k\ \ l)c_{l,m}$ *for all distinct* k, l, m.

To denote products of k disjoint contractions we introduce the following notation. Suppose $\underline{m} = (m_1, ..., m_k)$ and $\underline{n} = (n_1, ..., n_k)$ are ordered subsets of $\mathscr{P}$ and $\mathscr{Q}$, respectively, and let $(\underline{m}, \underline{n}) = \{(m_1, n_1), ..., (m_k, n_k)\}$. Let $P(0) = \emptyset$, and for $k \geqslant 1$ define

$$P(k) = \{(\underline{m}, \underline{n}) \mid \underline{m} \subseteq \mathscr{P},\ \underline{n} \subseteq \mathscr{Q}, \text{ and } |\underline{m}| = k = |\underline{n}|\}$$

and

$$P = \bigcup_{k=0}^{\min(p,q)} P(k).$$

For $(\underline{m}, \underline{n}) \in P(k)$, the corresponding *k-contraction* $c_{\underline{m},\underline{n}}$ is given by

$$c_{\underline{m},\underline{n}} = c_{m_1,n_1} \cdots c_{m_k,n_k} \qquad \text{if} \quad k \geqslant 1,$$

and

$$c_{\emptyset,\emptyset} = \mathrm{id} \qquad \text{if} \quad k = 0.$$

Suppose that $\mathscr{K} \subseteq \mathscr{P}$ and $\mathscr{L} \subseteq \mathscr{Q}$, and let $\tau = [\tau^+, \tau^-] \in \mathscr{T}(\mathscr{K}, \mathscr{L})$. To τ^+ we associate two subgroups in the group $S_{\mathscr{K}}$, which we view as a subgroup of both $S_{\mathscr{P}}$ and $S_{\mathscr{P}} \times S_{\mathscr{Q}}$. The first is the *row group* $\mathscr{R}_{\tau^+}$ consisting of all permutations in $S_{\mathscr{K}}$ which transform each entry of τ^+ to an entry in the same row, while the second is the *column group* $\mathscr{C}_{\tau^+}$ of τ^+ or group of permutations in $S_{\mathscr{K}}$ moving only the entries of τ^+ within each column. Using these subgroups, we construct the following element in the group algebra $\mathbb{C}[S_{\mathscr{P}} \times S_{\mathscr{Q}}]$:

$$s_{\tau^+} = \Big(\sum_{\gamma \in \mathscr{C}_{\tau^+}} \operatorname{sgn}(\gamma)\gamma\Big)\Big(\sum_{\psi \in \mathscr{R}_{\tau^+}} \psi\Big) = \sum_{\substack{\psi \in \mathscr{R}_{\tau^+} \\ \gamma \in \mathscr{C}_{\tau^+}}} \operatorname{sgn}(\gamma)\gamma\psi. \tag{2.2}$$

Similarly for τ^- we associate $\mathscr{R}_{\tau^-}$ and $\mathscr{C}_{\tau^-}$ in $S_{\mathscr{L}} \subseteq S_{\mathscr{Q}} \subseteq S_{\mathscr{P}} \times S_{\mathscr{Q}}$, and construct $s_{\tau^-} \in \mathbb{C}[S_{\mathscr{P}} \times S_{\mathscr{Q}}]$.

There exist integers $a, b \in \mathbb{Z}^+$ such that $s_{\tau^+}^2 = as_{\tau^+}$ and $s_{\tau^-}^2 = bs_{\tau^-}$ (see [Wey, p. 121]), so that $y_{\tau^+} = (1/a)s_{\tau^+}$ and $y_{\tau^-} = (1/b)s_{\tau^-}$ are commuting idempotents. The *Young symmetrizer* for τ is the idempotent $y_\tau = y_{\tau^+} y_{\tau^-}$.

We say the row group of τ is $\mathscr{R}_\tau = \mathscr{R}_{\tau^+} \times \mathscr{R}_{\tau^-}$ and the column group is $\mathscr{C}_\tau = \mathscr{C}_{\tau^+} \times \mathscr{C}_{\tau^-}$, which are subgroups of $S_{\mathscr{P}} \times S_{\mathscr{Q}}$. Then

$$y_\tau = (1/ab)\left(\sum_{\gamma \in \mathscr{C}_\tau} \operatorname{sgn}(\gamma)\gamma\right)\left(\sum_{\psi \in \mathscr{R}_\tau} \psi\right) = (1/ab) \sum_{\substack{\psi \in \mathscr{R}_\tau \\ \gamma \in \mathscr{C}_\tau}} \operatorname{sgn}(\gamma)\gamma\psi.$$

When τ and τ' are two standard rational tableaux in $\mathscr{S}\mathscr{T}(\mathscr{K}, \mathscr{L})$ having same underlying frame, we introduce an order which compares the entries of τ and τ'. We start at the left end of the first rows of τ^+ and τ'^+ and move from left to right. If the first nonzero difference $m - m'$ is positive for corresponding entries m in τ^+ and m' in τ'^+, then we say $\tau > \tau'$. If all the corresponding entries in the first rows are equal, then we move to the second rows, etc. If τ^+ and τ'^+ are identical, then we proceed with τ^- and τ'^- in the same fashion. With respect to this order we have the following lemma which is an immediate consequence of [BBL, Proposition 2.3] and the commutativity of $y_{\tau'^-}$ and y_{τ^+}.

LEMMA 2.3. *Let $\tau, \tau' \in \mathscr{S}\mathscr{T}(\mathscr{K}, \mathscr{L})$ be two standard rational tableaux associated to the same frame. If $\tau > \tau'$, then $y_{\tau'} y_\tau = 0$.*

In Theorem 1.11 we have seen that the irreducible summands of T are in one-to-one correspondence with pairs (τ, L) composed of a standard rational tableau τ and a two line array

$$L = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}$$

which satisfy (1.10). Each pair a_i, b_i consists of an element m_i of $\mathscr{P}$ and an element n_i of $\mathscr{Q}$. Thus we can identify L with $(\underline{m}, \underline{n}) \in P(k)$, where $\underline{m} = (m_1, ..., m_k)$ and $\underline{n} = (n_1, ..., n_k)$. Since the entries in τ and L are disjoint, we have $\tau \in \mathscr{S}\mathscr{T}(\underline{m}^c, \underline{n}^c)$, where $\underline{m}^c, \underline{n}^c$ are the complements of $\underline{m}, \underline{n}$ in $\mathscr{P}$ and $\mathscr{Q}$, respectively. Let y_τ be the Young symmetrizer associated to τ and $c_{\underline{m},\underline{n}}$ the k-contraction map corresponding to $(\underline{m}, \underline{n})$. Our next goal is to exhibit a maximal vector in T corresponding to each pair $(y_\tau, c_{\underline{m},\underline{n}})$.

A *maximal vector* t for $\mathscr{G}$ is a common eigenvector for the Cartan subalgebra $\mathscr{H}$ of diagonal matrices and is annihilated by the strictly upper triangular matrices in $\mathscr{G}$. Let ε_i denote the projection of a matrix onto its (i, i) entry as before. Then the weight of each basis vector v_j relative to $\mathscr{H}$ is ε_j and the weight of v_j^* is $-\varepsilon_j$. Moreover, if $t = v_{i_1} \otimes \cdots \otimes v_{i_p} \otimes v_{i_{p+1}}^* \otimes \cdots \otimes v_{i_{p+q}}^*$, the weight of t is $\eta_1 \varepsilon_1 + \cdots + \eta_r \varepsilon_r$, where η_j is the number of v_{i_k} equal to v_j minus the number of $v_{i_l}^*$ equal to v_j^*. Consequently, the weights of the simple tensors in T are integral combinations of $\varepsilon_1, ..., \varepsilon_r$. Each irreducible $\mathscr{G}$-module possesses a unique maximal vector up to scalar multiple whose weight $\lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r$ determines an r-staircase $\lambda = \{\lambda_1 \geqslant \cdots \geqslant \lambda_r\}$. Since an irreducible module is generated by any nonzero vector, in particular its maximal vector, locating the maximal vectors in T enables us to find its irreducible summands.

DEFINITION 2.4. Asumme $(\underline{m}, \underline{n}) \in P(k)$ for some k, and suppose $\tau \in \mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$, where $[\mu, \nu] \in \Pi(p-k, q-k)$. For $r \geqslant \max(l(\mu), l(\nu))$,

$$\beta_{\tau,\underline{m},\underline{n}} = u_1 \otimes \cdots \otimes u_p \otimes u^*_{p+1} \otimes \cdots \otimes u^*_{p+q}$$

is the simple tensor whose factors are defined by

$$u_k = \begin{cases} v_1 & \text{if } k \in \underline{m} \\ v_j & \text{if } k \in \underline{m}^c \text{ and } k \text{ in the } j\text{th row of } \tau^+, \end{cases}$$

$$u^*_k = \begin{cases} v^*_1 & \text{if } k \in \underline{n} \\ v^*_{r-j+1} & \text{if } k \in \underline{n}^c \text{ and } k \text{ in the } j\text{th row of } \tau^-, \end{cases}$$

and $t_{\tau,\underline{m},\underline{n}} =^{\text{def}} y_\tau c_{\underline{m},\underline{n}} \beta_{\tau,\underline{m},\underline{n}}$.

EXAMPLE. Suppose $p = 3$ and $q = 4$, and let $(\underline{m}, \underline{n}) = \{(1, 6)\}$ and

$$\tau = \left[\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 7 \\ \hline 5 & \\ \cline{1-1} \end{array} \right].$$

Then $\beta_{\tau,\underline{m},\underline{n}} = v_1 \otimes v_1 \otimes v_2 \otimes v^*_r \otimes v^*_{r-1} \otimes v^*_1 \otimes v^*_r$, $y_{\tau^+} = 1/2(\text{id} - (2\ 3))$, and $y_{\tau^-} = 1/3((\text{id} - (4\ 5))(\text{id} + (4\ 7)))$. Calculation of $t_{\tau,\underline{m},\underline{n}} = y_\tau c_{\underline{m},\underline{n}} \beta_{\tau,\underline{m},\underline{n}}$ shows

$$\begin{aligned} t_{\tau,\underline{m},\underline{n}} = \tfrac{1}{3} \sum_{i=1}^{r} \bigl(& v_i \otimes v_1 \otimes v_2 \otimes v^*_r \otimes v^*_{r-1} \otimes v^*_i \otimes v^*_r \\ & - v_i \otimes v_1 \otimes v_2 \otimes v^*_{r-1} \otimes v^*_r \otimes v^*_i \otimes v^*_r \\ & + v_i \otimes v_2 \otimes v_1 \otimes v^*_{r-1} \otimes v^*_r \otimes v^*_i \otimes v^*_r \\ & - v_i \otimes v_2 \otimes v_1 \otimes v^*_r \otimes v^*_{r-1} \otimes v^*_i \otimes v^*_r \bigr). \end{aligned}$$

THEOREM 2.5. *For $(\underline{m}, \underline{n}) \in P(k)$, $\tau \in \mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$, and all $r \geqslant \max(l(\mu), l(\nu))$, $t_{\tau,\underline{m},\underline{n}} = y_\tau c_{\underline{m},\underline{n}} \beta_{\tau,\underline{m},\underline{n}}$ is a maximal vector of $T = (\otimes^p V) \otimes (\otimes^q V^*)$. Moreover, relative to $\mathcal{H}$ the weight of $t_{\tau,\underline{m},\underline{n}}$ is $\mu_1\varepsilon_1 + \cdots + \mu_{l(\mu)}\varepsilon_{l(\mu)} - \nu_{l(\nu)}\varepsilon_{r-l(\nu)+1} - \cdots - \nu_1\varepsilon_r$.*

To prove Theorem 2.5 we require the following lemma:

LEMMA 2.6. *Let $t = t_1 \otimes \cdots \otimes t_{p+q} \in T$ and suppose τ is a rational tableau. If $(k\ l) \in \mathscr{C}_\tau$ for $k \neq l$ and if $t_k = t_l$, then $\sum_{\gamma \in \mathscr{C}_\tau} \text{sgn}(\gamma)\gamma t = 0$.*

Proof. $\sum_{\gamma \in \mathscr{C}_\tau} \operatorname{sgn}(\gamma)\gamma t = \sum_{\gamma \in \mathscr{C}_\tau} \operatorname{sgn}(\gamma(k\ l))\, \gamma(k\ l) t = -\sum_{\gamma \in \mathscr{C}_\tau} \operatorname{sgn}(\gamma)\gamma t.$ ∎

Proof of Theorem 2.5. Let $\beta = \beta_{\tau,\underline{m},\underline{n}}$. Then $\psi\beta = \beta$ for all $\psi \in \mathscr{R}_\tau$, and for $\gamma \in \mathscr{C}_\tau$ we have $\gamma\beta = \beta$ if and only if $\gamma = \mathrm{id}$. Thus, when $t_{\tau,\underline{m},\underline{n}}$ is expressed as a linear combination of simple tensors, the coefficient of β is the order of $\mathscr{R}_\tau$, and therefore $t_{\tau,m,n} \neq 0$. Since the weights of v_i and v_i^* sum to zero, the weight of $t_{\tau,\underline{m},\underline{n}}$ is the same as that of β, which is $\mu_1\varepsilon_1 + \cdots + \mu_{l(\mu)}\varepsilon_{l(\mu)} - \nu_{l(\nu)}\varepsilon_{r-l(\nu)+1} - \cdots - \nu_1\varepsilon_r$ by the way β is defined. To prove that $t_{\tau,\underline{m},\underline{n}}$ is maximal, it suffices to show that it is annihilated by $x_j = E_{j,j+1}$ for $j = 1, ..., r-1$, since the matrix units $x_j = E_{j,j+1}$ generate the subalgebra of strictly upper triangular matrices. Now $x_j \cdot v_{j+1} = v_j$ and $x_j \cdot v_j^* = -v_{j+1}^*$, while $x_j \cdot v_k = 0$ for $k \neq j+1$ and $x_j \cdot v_l^* = 0$ for $l \neq j$. The vector $\sum_{i=1}^r v_i \otimes v_i^*$ spans a trivial $\mathscr{G}$-submodule, hence is annihilated by each x_j. Thus, $x_j \cdot c_{\underline{m},\underline{n}}\beta = 0$ or $x_j \cdot c_{\underline{m},\underline{n}}\beta$ is a sum of simple tensors each having exactly one v_{j+1} changed to v_j or one v_j^* changed to $-v_{j+1}^*$ in a noncontracted slot of $c_{\underline{m},\underline{n}}\beta$. For such a simple tensor t, we argue that $y_\tau t = 0$. Since τ is a standard tableau, if v_{j+1} is changed to v_j in slot k, there must be a v_j in some slot $l \neq k$, where $(k\ l)$ is in the column group $\mathscr{C}_\tau$. Likewise, if v_j^* is changed to $-v_{j+1}^*$ in slot k, there must be a v_{j+1}^* in slot $l \neq k$, where $(k\ l)$ is in the column group $\mathscr{C}_\tau$. Thus, by Lemma 2.6, $\sum_{\gamma \in \mathscr{C}_\tau} \operatorname{sgn}(\gamma)\gamma\psi t = \sum_{\gamma \in \mathscr{C}_\tau} \operatorname{sgn}(\gamma)\gamma t = 0$ for each ψ in the row group $\mathscr{R}_\tau$. Hence $x_j \cdot y_\tau c_{\underline{m},\underline{n}}\beta = y_\tau(x_j \cdot c_{\underline{m},\underline{n}}\beta) = 0$. ∎

We now address the question of linear independence of these maximal vectors.

THEOREM 2.7. *The following are linearly independent sets of maximal vectors for the listed values of* r:

$$\{t_{\tau,\underline{m},\underline{n}} = y_\tau c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}} \mid (\underline{m}, \underline{n}) \in P(k),\ \tau \in \mathscr{ST}(\underline{m}^c, \underline{n}^c),$$
$$\tau \textit{ shape } [\mu, \nu] \in \Pi(p-k, q-k)\}$$
$$\textit{for all} \quad r \geqslant l(\mu) + l(\nu) + k,$$

$$\{t_{\tau,\underline{m},\underline{n}} = y_\tau c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}} \mid (\underline{m}, \underline{n}) \in P(k),\ \tau \in \mathscr{ST}(\underline{m}^c, \underline{n}^c)\}$$
$$\textit{for all} \quad r \geqslant p + q - k,$$

$$\{t_{\tau,\underline{m},\underline{n}} = y_\tau c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}} \mid (\underline{m}, \underline{n}) \in P,\ \tau \in \mathscr{ST}(\underline{m}^c, \underline{n}^c)\}$$
$$\textit{for all} \quad r \geqslant p + q.$$

Proof. Vectors having different numbers of contractions are linearly independent because those vectors have different weights. Similarly if the rational tableaux have different shapes, then the maximal vectors have different weights and are linearly independent. Thus, it suffices to prove the first assertion and the remainder will follow. Suppose

$$\sum_{(\underline{m},\underline{n})\in P(k)} \sum_{\tau\in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\underline{n}^c)} a_{\tau,\underline{m},\underline{n}}\, y_\tau c_{\underline{m},\underline{n}} \beta_{\tau,\underline{m},\underline{n}} = 0. \tag{2.8}$$

If $k \geqslant 1$, then among the simple tensors summing up to $c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}}$ there is a unique tensor $\xi_{\tau,\underline{m},\underline{n}}$ having $v_{l(\mu)+j}$ as the m_jth factor and $v^*_{l(\mu)+j}$ as the n_jth factor for each $j = 1, ..., k$. If $k = 0$, we set $\xi_{\tau,\emptyset,\emptyset} = c_{\emptyset,\emptyset}\beta_{\tau,\emptyset,\emptyset}$. We note that $l(\mu)+k \leqslant (p-k)+k \leqslant r$, so the vectors $v_{l(\mu)+j}$ and $v^*_{l(\mu)+j}$ have their indices in proper range. We also observe that $l(\mu)+k < r-l(\nu)+1$, since $r \geqslant l(\mu)+l(\nu)+1$ by assumption. Thus the factors $v_{l(\mu)+j}$ and $v^*_{l(\mu)+j}$ for $j = 1, ..., k$ are distinct from all the factors in $\beta_{\tau,\underline{m},\underline{n}}$. Then for every k it follows that

$$\sum_{(\underline{m},\underline{n})\in P(k)} \sum_{\tau\in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\underline{n}^c)} a_{\tau,\underline{m},\underline{n}}\, y_\tau \xi_{\tau,\underline{m},\underline{n}} = 0. \tag{2.9}$$

The simple tensors in (2.9) corresponding to different choices of $(\underline{m}, \underline{n})$ are linearly independent, as the factors $v_{l(\mu)+j}$ and $v^*_{l(\mu)+j}$ are sufficient to distinguish them. Thus, for any $(\underline{m}, \underline{n}) \in P(k)$,

$$\sum_{\tau\in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\underline{n}^c)} a_{\tau,\underline{m},\underline{n}}\, y_\tau \xi_{\tau,\underline{m},\underline{n}} = 0. \tag{2.10}$$

If some $a_{\tau,\underline{m},\underline{n}} \neq 0$, then we may choose τ' minimal with that property. By Lemma 2.3 $\tau > \tau'$ implies that $y_{\tau'} y_\tau = 0$. Thus applying $y_{\tau'}$ to (2.10), we obtain

$$0 = a_{\tau',\underline{m},\underline{n}}\, y_{\tau'} y_{\tau'} \xi_{\tau',\underline{m},\underline{n}} = a_{\tau',\underline{m},\underline{n}}\, y_{\tau'} \xi_{\tau',\underline{m},\underline{n}}.$$

Since $\xi_{\tau',\underline{m},\underline{n}}$ appears in $y_{\tau'}\xi_{\tau',\underline{m},\underline{n}}$ with coefficient equal to the order of the row group of τ', then $a_{\tau',\underline{m},\underline{n}} = 0$. Hence no such τ' exists, and $a_{\tau,\underline{m},\underline{n}} = 0$ for all τ and all $(\underline{m}, \underline{n})$. ∎

Hanlon [Han1, Han2] has used the theory of Specht modules for the symmetric group to describe the maximal vectors in the tensor product $\otimes^p \mathscr{G}$ of the adjoint representation of $\mathscr{G}$. Since $V \otimes V^* \cong \mathscr{G}$ as $\mathscr{G}$-modules, Hanlon's results [Han2, Theorems 4.12 and 4.13] could be recast to give the maximal vectors in $(\otimes^p V) \otimes (\otimes^p V^*) \cong \otimes^p(V \otimes V^*)$ and their independence for r sufficiently large. The approach adopted above allows us to treat the case of arbitrary p and q uniformly and reduces to a somewhat easier (though equivalent) description of the maximal vectors in the $p = q$ case. The maximal vectors for the $q = 0$ case have been constructed in [BBL, Theorem 2.33 and Lemma 2.39].

Combining Theorems 2.5 and 2.7 we have

THEOREM 2.11. *Assume $r \geqslant p+q$ and let $\mathcal{U}(\mathcal{G})$ denote the universal enveloping algebra of $\mathcal{G} = gl(r, \mathbb{C})$. Then*

$$T = \bigoplus \sum_{(\underline{m},\underline{n}) \in P} \sum_{\tau \in \mathcal{ST}(\underline{m}^c, \underline{n}^c)} \mathcal{U}(\mathcal{G}) t_{\tau,\underline{m},\underline{n}}$$

$$= \bigoplus \sum_{(\underline{m},\underline{n}) \in P} \sum_{\tau \in \mathcal{ST}(\underline{m}^c, \underline{n}^c)} \mathcal{U}(\mathcal{G}) y_\tau c_{\underline{m},\underline{n}} \beta_{\tau,\underline{m},\underline{n}}$$

is the decomposition of T into irreducible $\mathcal{G}$-modules.

Theorem 2.11 can be phrased in the language of partitions. In doing so, it is convenient to use the notion of the hook length of a partition $\pi = \{\pi_1 \geqslant \cdots \geqslant \pi_{l(\pi)} > 0\} \in \Pi(N)$. The *hook length*, $h(i, j)$, of the (i, j) box of $\mathcal{F}(\pi)$ is defined by $h(i, j) = \pi_i + \pi_j^* - i - j + 1$, where π_j^* is the number of boxes in the jth column of π. The expression $\pi_i - i$ counts the number of boxes in the ith row of $\mathcal{F}(\pi)$ to the right of (i, j), while $\pi_j^* - j$ counts the number of boxes in the jth column of $\mathcal{F}(\pi)$ below (i, j). Therefore, $h(i, j)$ is simply the number of boxes to the right and below (i, j) plus 1 to include the (i, j)th box. The *hook length of* π, $h(\pi)$, is the product over all (i, j) of the hook lengths $h(i, j)$. The number of standard tableaux having underlying frame $\mathcal{F}(\pi)$ is given by the expression $N!/h(\pi)$ (see [FRT] or [J, p. 77]). For ease of notation we adopt the convention that $h(\emptyset) = 1$. We now have necessary ingredients for restating Theorem 2.11.

THEOREM 2.12. *If $r \geqslant p+q$, then*

$$T \cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,q-k)} \frac{p!\,q!}{k!\,h(\mu)\,h(\nu)} V([\mu, \nu]_r),$$

where $V([\mu, \nu]_r)$ denotes the finite dimensional irreducible $\mathcal{G}$-module with highest weight $\lambda = \mu_1\varepsilon_1 + \cdots + \mu_{l(\mu)}\varepsilon_{l(\mu)} - \nu_{l(\nu)}\varepsilon_{r-l(\nu)+1} - \cdots - \nu_1\varepsilon_r$ corresponding to the r-staircase $\lambda = [\mu, \nu]_r$.

Proof. By Theorem 2.11 the irreducible summands are in one-to-one correspondence with the pairs $(\tau, (\underline{m}, \underline{n}))$, where $(\underline{m}, \underline{n}) \in P(k)$ for some k and $\tau \in \mathcal{ST}(\underline{m}^c, \underline{n}^c)$, and the summand corresponding to $(\tau, (\underline{m}, \underline{n}))$ is isomorphic to $V([\mu, \nu]_r)$, where $[\mu, \nu] \in \Pi(p-k, q-k)$ is the shape of τ. For a fixed k with $0 \leqslant k \leqslant \min(p, q)$, the number of distinct pairs $(\underline{m}, \underline{n}) \in P(k)$ is $\binom{p}{k}\binom{q}{k}k!$, where $k!$ counts the number of different ways to pair k objects with k other objects. For $\tau \in \mathcal{ST}(\underline{m}^c, \underline{n}^c)$, the entries in τ^+ and in τ^- are distinct. Thus, the number of standard rational tableaux τ of shape $[\mu, \nu] \in \Pi(p-k, q-k)$ is the product of the number of standard tableaux

τ^+ of shape $\mu \in \Pi(p-k)$, which is $(p-k)!/h(\mu)$, with the number of standard tableaux τ^- of shape $\nu \in \Pi(q-k)$, which is $(q-k)!/h(\nu)$. Therefore, we have

$$T \cong \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,q-k)} \binom{p}{k}\binom{q}{k} k! \frac{(p-k)!}{h(\mu)} \frac{(q-k)!}{h(\nu)} V([\mu,\nu]_r)$$
$$\cong \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,q-k)} \frac{p!\,q!}{k!\,h(\mu)\,h(\nu)} V([\mu,\nu]_r). \quad \blacksquare$$

For $[\mu, \nu] \in \Pi(p-k, q-k)$, let $z_{\mu,\nu}$ denote the multiplicity of $V([\mu,\nu]_r)$ in the decomposition of T into irreducible $\mathscr{G}$-modules, and let

$$(2.13) \qquad m_{\mu,\nu} \stackrel{\text{def}}{=} \binom{p}{k}\binom{q}{k} k! \frac{(p-k)!}{h(\mu)} \frac{(q-k)!}{h(\nu)} = \frac{p!\,q!}{k!\,h(\mu)\,h(\nu)}.$$

Then by Theorem 2.12, $z_{\mu,\nu} = m_{\mu,\nu}$ for all $[\mu, \nu] \in \Pi(p-k, q-k)$ when $r \geqslant p+q$, and the decomposition of $(\bigotimes^p V) \otimes (\bigotimes^q V^*)$ is the same for all $gl(r, \mathbb{C})$ with $r \geqslant p+q$ provided the pairs of partitions (r-staircases) are interpreted appropriately. Without the rank assumption $r \geqslant p+q$ we have

THEOREM 2.14 (Compare [St2], Proposition 4.8). *For all* $[\mu, \nu] \in \Pi(p-k, q-k)$, $z_{\mu,\nu} \leqslant m_{\mu,\nu}$. *Moreover,* $z_{\mu,\nu} = m_{\mu,\nu}$ *if and only if* $r \geqslant l(\mu) + l(\nu) + k$. *In particular, if* $r \geqslant p+q-k$, *then* $z_{\mu,\nu} = m_{\mu,\nu}$ *for all* $[\mu, \nu] \in \Pi(p-k, q-k)$.

Proof. By Theorem 1.11 the irreducible summands of T isomorphic to $V([\mu,\nu]_r)$ are in one-to-one correspondence with the pairs (τ, L) or $(\tau, (\underline{m}, \underline{n}))$ having τ of shape $[\mu, \nu]$ and corresponding to r-staircases. Our calculations in the proof of Theorem 2.12 show that the number of those pairs in which τ has shape $[\mu, \nu]$ is $m_{\mu,\nu}$. Thus, $z_{\mu,\nu} \leqslant m_{\mu,\nu}$. If $r \geqslant l(\mu) + l(\nu) + k$, then by Theorem 2.7 the maximal vectors

$$\{t_{\tau,m,n} = y_\tau c_{\underline{m},\underline{n}} \beta_{\tau,m,n} \mid (\underline{m}, \underline{n}) \in P(k),\ \tau \in \mathscr{S}\mathscr{T}(\underline{m}^c, \underline{n}^c),$$
$$\tau \text{ shape } [\mu, \nu] \in \Pi(p-k, q-k)\}$$

are linearly independent. Thus, $z_{\mu,\nu} = m_{\mu,\nu}$ for all $r \geqslant l(\mu) + l(\nu) + k$. To see that the rank restriction is necessary let $[\mu, \nu] \in \Pi(p-k, q-k)$ with $r < l(\mu) + l(\nu) + k$. Note that if $k=0$, then $r < l(\mu) + l(\nu)$ and $[\mu, \nu]_r$ is not an r-staircase, so we may assume that $k > 0$. Let $\mathscr{K} = \{1, ..., |\mu|\}$, $\mathscr{L} = \{p+1, ..., p+|\nu|\}$, and suppose $\tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\mathscr{K}, \mathscr{L})$. Let L be the following two line array.

$$L = \begin{pmatrix} p+q-k+1 & p+q-k+2 & \cdots & p+q-1 & p+q \\ p & p-1 & \cdots & |\mu|+2 & |\mu|+1 \end{pmatrix}.$$

Then for $0 \leqslant j \leqslant k-1$, step $p+q-j$ of algorithm $\mathfrak{B}$ applied to (τ, L) inserts $|\mu|+j+1$ into τ^+. Since each entry of the bottom row of L is greater than the largest entry of τ^+, and the entries in the bottom row increase from right to left, steps $p+q, ..., p+q-k+1$ of algorithm $\mathfrak{B}$ result in a rational tableau having a total of $l(\mu)+l(\nu)+k > r$ rows. This does not correspond to an r-staircase. Thus (τ, L) does not index an irreducible $\mathscr{G}$-submodule of T. ∎

Remark. We present an example below to illustrate the stability of the decomposition for all $r \geqslant p+q$ and to show how the result can be modified for small values of r in accordance with Theorems 1.11 and 2.14. Here it is notationally somewhat easier to revert to the r-staircase notation, so that the summand $V([\{3\}, \{1^2\}]_r)$ corresponding to the pair of partitions $[\{3\}, \{1^2\}]$ will be denoted $V(3, 0^{r-3}, (-1)^2)$.

EXAMPLE 2.15. Consider $T = (\otimes^3 V) \otimes (\otimes^2 V^*)$ for $\mathscr{G} = gl(r, \mathbb{C})$. Then by Theorem 2.12 we have for $r \geqslant p+q=5$

$$(r \geqslant 5) \qquad \begin{aligned} T \cong\ & V(3, 0^{r-2}, -2) \oplus V(3, 0^{r-3}, (-1)^2) \\ & \oplus 2V(2, 1, 0^{r-3}, -2) \oplus 2V(2, 1, 0^{r-4}, (-1)^2) \\ & \oplus V(1^3, 0^{r-4}, -2) \oplus V(1^3, 0^{r-5}, (-1)^2) \\ & \oplus 6V(2, 0^{r-2}, -1) \oplus 6V(1^2, 0^{r-3}, -1) \oplus 6V(1, 0^{r-1}). \end{aligned}$$

The first three lines on the right side correspond to the case $k=0$, the next two terms to $k=1$, and the final term to $k=2$. Now when $r < p+q$ we can apply Theorems 1.11 and 2.14 to determine the decomposition. Every pair $(\tau, (\underline{m}, \underline{n}))$ in Theorem 2.11, which when converted to an up-down r-staircase tableau produces a staircase which is too long, must be removed from the sum. When $k=0$, we see that the sixth summand must be omitted for $r=4$, the fourth through the sixth for $r=3$, and the second through the sixth for $r=2$. By Theorem 2.14 the remaining summands in the first three rows have the same multiplicities as in the stable case $(r \geqslant 5)$. When $k=1$, the summand corresponding to $\tau = \left[\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}, \begin{array}{|c|}\hline 4\\ \hline\end{array}\right]$ and $(\underline{m}, \underline{n}) = \{(3, 5)\}$ (or $L = ({}^5_3)$) must be deleted if $r=3$ because in the conversion process the staircase $\lambda^4 = (1^3, -1)$ is produced when the number 3 is inserted into τ. Similarly the summands corresponding to $\tau = \left[\begin{array}{|c|c|}\hline 1 & 3\\ \hline\end{array}, \begin{array}{|c|}\hline 4\\ \hline\end{array}\right]$ and $(\underline{m}, \underline{n}) = \{(2, 5)\}$, $\tau = \left[\begin{array}{|c|c|}\hline 1 & 2\\ \hline\end{array}, \begin{array}{|c|}\hline 4\\ \hline\end{array}\right]$ and $(\underline{m}, \underline{n}) = \{(3, 5)\}$, and all τ of shape $[\{1^2\}, \{1\}]$ must be removed when $r=2$. Finally, when $k=2$, the summand associated to the pair $\tau = \left[\boxed{\;1\;}, \emptyset\right]$, $(\underline{m}, \underline{n}) = \{(3,4), (2,5)\}$ is omitted for $r=2$. Thus, we have

$$(r=4)\qquad \begin{aligned} T \cong\; & V(3, 0^2, -2) \oplus V(3, 0, (-1)^2) \oplus 2V(2, 1, 0, -2) \\ & \oplus 2V(2, 1, (-1)^2) \oplus V(1^3, -2) \\ & \oplus 6V(2, 0^2, -1) \oplus 6V(1^2, 0, -1) \oplus 6V(1, 0^3) \end{aligned}$$

$$(r=3)\qquad \begin{aligned} T \cong\; & V(3, 0, -2) \oplus V(3, (-1)^2) \oplus 2V(2, 1, -2) \\ & \oplus 6V(2, 0, -1) \oplus 5V(1^2, -1) \oplus 6V(1, 0^2) \end{aligned}$$

$$(r=2)\qquad T \cong V(3, -2) \oplus 4V(2, -1) \oplus 5V(1, 0).$$

The results in the low rank cases could also be derived from the modification rules which appear in [Ki].

3. The Action of $S_{\mathscr{P}} \times S_{\mathscr{Q}}$ and $c_{m,n}$ on the Space $M_{\mu,\nu}$ of Maximal Vectors

For the Lie algebra $\mathscr{G} = gl(r, \mathbb{C})$ and the $\mathscr{G}$-module $T = (\otimes^p V) \otimes (\otimes^q V^*)$, the *centralizer algebra* of $\mathscr{G}$ on T is the set

$$\mathscr{C} = \operatorname{End}_{\mathscr{G}}(T) = \{\phi \in \operatorname{End}_{\mathbb{C}}(T) \mid \phi(x \cdot t) = x \cdot \phi(t) \text{ for all } x \in \mathscr{G} \text{ and } t \in T\}$$

of linear transformations on T which commute with $\mathscr{G}$. Since $\mathscr{C}$ consists of all $\mathscr{G}$-module endomorphisms of T, the projection maps of T onto its irreducible $\mathscr{G}$-summands lie in $\mathscr{C}$. As we have seen in Section 2, the permutations in $S_{\mathscr{P}} \times S_{\mathscr{Q}}$ and the contraction maps $c_{m,n}$ commute with $\mathscr{G}$, so the group algebra $\mathbb{C}[S_{\mathscr{P}} \times S_{\mathscr{Q}}]$ and the algebra generated by all contractions lie in $\mathscr{C}$. We define $\mathscr{A}$ to be the subalgebra of $\mathscr{C}$ generated by $S_{\mathscr{P}} \times S_{\mathscr{Q}}$ and the set $\{c_{m,n} \mid 1 \leqslant m \leqslant p \text{ and } p+1 \leqslant n \leqslant p+q\}$. In Section 5 we will show that $\mathscr{A} = \mathscr{C}$.

In this section we investigate the action of $\mathscr{A}$ on T and, in particular, on the maximal vectors in T of a given weight. In Section 1 we proved that the pairs $(\tau, (\underline{m}, \underline{n}))$ index the irreducible $\mathscr{G}$-summands of T. Fix $[\mu, \nu] \in \Pi(p-k, q-k)$ for some $k = 0, ..., \min(p, q)$ and let $M_{\mu,\nu}$ be the $\mathbb{C}$-vector space spanned by the maximal vectors of weight $\lambda = \mu_1\varepsilon_1 + \cdots + \mu_{l(\mu)}\varepsilon_{l(\mu)} - \nu_{l(\nu)}\varepsilon_{r-l(\nu)+1} - \cdots - \nu_1\varepsilon_r$. Then we set

$$B_{\mu,\nu} = \{t_{\tau,\underline{m},\underline{n}} \mid (\underline{m}, \underline{n}) \in P(k) \text{ and } \tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)\}$$

is a basis for $M_{\mu,\nu}$ provided $r \geqslant l(\mu)+l(\nu)+k$ by Theorem 2.7. When $r \geqslant l(\mu)+l(\nu)+k$ we call $B_{\mu,\nu}$ the *standard basis* of $M_{\mu,\nu}$, and we refer to the elements of $B_{\mu,\nu}$ as *standard basis elements*. The action of $\mathscr{C}$ on T preserves weights and the property of being a maximal vector, so $M_{\mu,\nu}$ is a module for $\mathscr{C}$ and hence for $\mathscr{A}$. In Section 4 we show that $M_{\mu,\nu}$ is an irreducible $\mathscr{A}$-module and that $M_{\mu',\nu'} \cong M_{\mu,\nu}$ if and only if $\mu'=\mu$ and $\nu'=\nu$. Here we obtain explicit information about the action of $\mathscr{A}$ on $B_{\mu,\nu}$.

Assume $\sigma \in S_{\mathscr{P}} \times S_{\mathscr{Q}}$, $(\underline{m}, \underline{n}) \in P(k)$, and $\tau \in \mathscr{ST}(\underline{m}^c, \underline{n}^c)$. Then σ acts on $\underline{m}$ and $\underline{n}$ by $\sigma \cdot \underline{m} = (\sigma(m_1), ..., \sigma(m_k))$ and $\sigma \cdot \underline{n} = (\sigma(n_1), ..., \sigma(n_k))$. We define $\sigma \cdot \tau$ to be the rational tableau obtained by applying σ to the entries of τ. The tableau $\sigma \cdot \tau$ need not be standard. However, from Definition 2.4 we have

$$\sigma\beta_{\tau,\underline{m},\underline{n}} = \beta_{\sigma\cdot\tau,\sigma\cdot\underline{m},\sigma\cdot\underline{n}}. \tag{3.1}$$

Therefore if $t_{\tau,\underline{m},\underline{n}} \in B_{\mu,\nu}$,

$$\begin{aligned} \sigma t_{\tau,\underline{m},\underline{n}} &= \sigma y_\tau \sigma^{-1}\sigma c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}} = y_{\sigma\cdot\tau} c_{\sigma\cdot\underline{m},\sigma\cdot\underline{n}}\beta_{\sigma\cdot\tau,\sigma\cdot\underline{m},\sigma\cdot\underline{n}} \\ &= t_{\sigma\cdot\tau,\sigma\cdot\underline{m},\sigma\cdot\underline{n}}. \end{aligned} \tag{3.2}$$

LEMMA 3.3. *The action of $S_{\mathscr{P}} \times S_{\mathscr{Q}}$ on $B_{\mu,\nu}$ is transitive.*

Proof. Assume $t_{\tau,\underline{m},\underline{n}}$ and $t_{\tau',\underline{m}',\underline{n}'}$ are in $B_{\mu,\nu}$. Since the entries of τ and the elements of $\underline{m}$ and $\underline{n}$ form a partition of $\mathscr{P} \cup \mathscr{Q}$, we can define $\sigma \in S_{\mathscr{P}} \times S_{\mathscr{Q}}$ by $\sigma(m_i)=m'_i$, $\sigma(n_i)=n'_i$, and $\sigma\cdot\tau=\tau'$. Then by (3.2), $\sigma t_{\tau,\underline{m},\underline{n}} = t_{\tau',\underline{m}',\underline{n}'}$ as desired. ∎

The next lemma describes the action of the contractions on the vectors $t_{\tau,\underline{m},\underline{n}}$.

LEMMA 3.4. *Assume $(\underline{m}, \underline{n}) = \{(m_1, n_1), ..., (m_k, n_k)\} \in P(k)$ and let $\tau \in \mathscr{T}(\underline{m}^c, \underline{n}^c)$. Then the contraction map $c_{a,b}$ acts on the vector $t_{\tau,\underline{m},\underline{n}}$ as follows:*

$$c_{a,b}t_{\tau,\underline{m},\underline{n}} = \begin{cases} 0 & \text{if (i) } a \in \underline{m}^c,\ b \in \underline{n}^c, \\ r t_{\tau,\underline{m},\underline{n}} & \text{if (ii) } (a,b) \in (\underline{m},\underline{n}), \\ \sigma t_{\tau,\underline{m},\underline{n}} = t_{\tau,\underline{m},\sigma\cdot\underline{n}} \text{ for } \sigma=(n_i n_j) & \\ & \text{if (iii) } a=m_i,\ b=n_j, \text{ and } i\neq j, \\ \sigma t_{\tau,\underline{m},\underline{n}} = t_{\sigma\cdot\tau,\sigma\cdot\underline{m},\underline{n}} \text{ for } \sigma=(m_i a) & \\ & \text{if (iv) } a\in\underline{m}^c \text{ and } b=n_i, \\ \sigma t_{\tau,\underline{m},\underline{n}} = t_{\sigma\cdot\tau,\underline{m},\sigma\cdot\underline{n}} \text{ for } \sigma=(n_i b) & \\ & \text{if (v) } a=m_i \text{ and } b\in\underline{n}^c. \end{cases}$$

Proof. The proof involves liberal use of the relations in Lemma 2.1. We illustrate by presenting the argument for (iv) and leave the rest to the reader.

(iv) We may suppose that $b=n_1$. For each $\gamma\in\mathscr{C}_\tau$ and $\psi\in\mathscr{R}_\tau$, the permutation $(\gamma\psi)^{-1}$ fixes m_1 and n_1. For notational convenience, let $\rho=\gamma\psi$ and $\sigma=(am_1)$. Then

$$\begin{aligned}
c_{a,n_1}\rho c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}} &= \rho c_{\rho^{-1}(a),n_1}c_{m_1,n_1}\cdots c_{m_k,n_k}\beta_{\tau,\underline{m},\underline{n}}\\
&=\rho(\rho^{-1}(a)m_1)\,c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}}\\
&=\sigma\rho c_{m_1,n_1}\cdots c_{m_k,n_k}\beta_{\tau,\underline{m},\underline{n}}=\sigma\rho c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
c_{a,n_1}t_{\tau,\underline{m},\underline{n}} &= \sum_{\substack{\psi\in\mathscr{R}_\tau\\ \gamma\in\mathscr{C}_\tau}} \operatorname{sgn}(\gamma)\,\sigma\gamma\psi c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}}\\
&=\sigma\sum_{\substack{\psi\in\mathscr{R}_\tau\\ \gamma\in\mathscr{C}_\tau}} \operatorname{sgn}(\gamma)\,\gamma\psi\sigma^{-1}\sigma c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}}\\
&=\sigma y_\tau\sigma^{-1}c_{\sigma\cdot\underline{m},\underline{n}}\sigma\beta_{\tau,\underline{m},\underline{n}}=t_{\sigma\cdot\tau,\sigma\cdot\underline{m},\underline{n}}. \quad \blacksquare
\end{aligned}$$

In both (3.2) and Lemma 3.4 it is possible that $\sigma\cdot\tau$ is not standard, and thus $t_{\sigma\cdot\tau,\sigma\cdot\underline{m},\sigma\cdot\underline{n}}$ is not a standard basis element. At the end of this section we present column permutation relations and Garnir relations that enable us to rewrite $t_{\sigma\cdot\tau,\sigma\cdot\underline{m},\sigma\cdot\underline{n}}$ as a $\mathbb{C}$-linear combination of the standard basis $B_{\mu,\nu}$, where $[\mu,\nu]$ is the shape of τ. However, before tackling that, we observe that Lemmas 2.1 and 3.4 can be extended by induction to yield the following results for k-contractions:

LEMMA 3.5. *Suppose that* $(\underline{a},\underline{b})$ *and* $(\underline{m},\underline{n})$ *are in* $P(k)$. *Then either*

(i) $c_{\underline{a},\underline{b}}c_{\underline{m},\underline{n}}=zc_{i,j}$, *where* $z\in\mathscr{A}$, $i\in\underline{m}^{\mathrm{c}}$, *and* $j\in\underline{n}^{\mathrm{c}}$, *or*

(ii) $c_{\underline{a},\underline{b}}c_{\underline{m},\underline{n}}=r^s\sigma c_{\underline{m},\underline{n}}=r^s c_{\underline{a},\underline{b}}\sigma$ *for some* $s\in\mathbb{Z}^+\cup\{0\}$, *where* $\sigma\in S_{\mathscr{I}}\times S_{\mathscr{J}}$ *is such that* $\sigma\cdot\underline{m}=\underline{a}$, $\sigma\cdot\underline{n}=\underline{b}$, *and* σ *fixes* $(\underline{m}^{\mathrm{c}}\cap\underline{a}^{\mathrm{c}})\cup(\underline{n}^{\mathrm{c}}\cap\underline{b}^{\mathrm{c}})$ *pointwise.*

COROLLARY 3.6. *Suppose that* $(\underline{a},\underline{b})$ *and* $(\underline{m},\underline{n})$ *are in* $P(k)$. *Then for* $\tau\in\mathscr{T}(\underline{m}^{\mathrm{c}},\underline{n}^{\mathrm{c}})$ *one of the following holds:*

(i) $c_{\underline{a},\underline{b}}t_{\tau,\underline{m},\underline{n}}=0$,

(ii) $c_{\underline{a},\underline{b}}t_{\tau,\underline{m},\underline{n}}=r^s t_{\sigma\cdot\tau,\underline{a},\underline{b}}$ *for some* $s\in\mathbb{Z}^+\cup\{0\}$, *where* $\sigma\in S_{\mathscr{I}}\times S_{\mathscr{J}}$ *is such that* $\sigma\cdot\underline{m}=\underline{a}$, $\sigma\cdot\underline{n}=\underline{b}$, *and* σ *fixes* $(\underline{m}^{\mathrm{c}}\cap\underline{a}^{\mathrm{c}})\cup(\underline{n}^{\mathrm{c}}\cap\underline{b}^{\mathrm{c}})$ *pointwise.*

Proof. If $c_{\underline{a},\underline{b}}c_{\underline{m},\underline{n}}=zc_{i,j}$, where $i\in\underline{m}^{\mathrm{c}}$ and $j\in\underline{n}^{\mathrm{c}}$, then by Lemma 3.4(i),

$$c_{\underline{a},\underline{b}}t_{\tau,\underline{m},\underline{n}}=zc_{i,j}t_{\tau,\underline{m},\underline{n}}=z\sum_{\substack{\psi\in\mathscr{R}_\tau\\ \gamma\in\mathscr{C}_\tau}}\operatorname{sgn}(\gamma)\,\gamma\psi c_{(\gamma\psi)^{-1}(i),(\gamma\psi)^{-1}(j)}c_{\underline{m},\underline{n}}\beta_{\tau,\underline{m},\underline{n}}=0.$$

If this is not the case, then by Lemma 3.5

$$\begin{aligned} c_{\underline{a},\underline{b}} t_{\tau,\underline{m},\underline{n}} &= c_{\underline{a},\underline{b}} c_{\underline{m},\underline{n}} y_\tau \beta_{\tau,\underline{m},\underline{n}} = r^s c_{\underline{a},\underline{b}} \sigma y_\tau \sigma^{-1} \sigma \beta_{\tau,\underline{m},\underline{n}} \\ &= r^s c_{\underline{a},\underline{b}} y_{\sigma\cdot\tau} \beta_{\sigma\cdot\tau,\sigma\cdot\underline{m},\sigma\cdot\underline{n}} = r^s t_{\sigma\cdot\tau,\underline{a},\underline{b}}, \end{aligned}$$

to give the desired conclusion. ∎

COROLLARY 3.7. *Assume that* $(\underline{m}, \underline{n}) \in P(k)$ *and* $(\underline{m}', \underline{n}') \in P(k')$, *where* $k' > k$. *Let* $[\mu, \nu] \in \Pi(p-k, q-k)$ *for some* $k = 0, \ldots, \min(p, q)$ *and suppose that* $\tau \in \mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$. *Then* $c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} = 0$, *and hence* $c_{\underline{m}',\underline{n}'}(M_{\mu,\nu}) = (0)$.

Proof. Suppose for some $(\underline{m}, \underline{n})$ there exists $\tau \in \mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$ with $c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} \neq 0$. Let $(\underline{m}', \underline{n}') = (\underline{a}', \underline{b}') \cup (\underline{a}, \underline{b})$, where $(\underline{a}, \underline{b}) \in P(k)$ and $(\underline{a}', \underline{b}') \in P(k'-k)$. Then since $c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} \neq 0$, we are in case (ii) of Corollary 3.6, and

$$c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} = c_{\underline{a}',\underline{b}'} c_{\underline{a},\underline{b}} t_{\tau,\underline{m},\underline{n}} = r^s c_{\underline{a}',\underline{b}'} t_{\sigma\cdot\tau,\underline{a},\underline{b}},$$

for some $s \in \mathbb{Z}^+ \cup \{0\}$ and some $\sigma \in S_{\mathscr{p}} \times S_{\mathscr{q}}$ with $\sigma \cdot \tau \in \mathcal{T}_{\mu,\nu}(\underline{a}^c, \underline{b}^c)$. Now $(\underline{a}', \underline{b}')$ is disjoint from $(\underline{a}, \underline{b})$, so by Lemma 3.4(i), $c_{\underline{a}',\underline{b}'} t_{\sigma\cdot\tau,\underline{a},\underline{b}} = 0$, which is a contradiction. ∎

We now turn to the problem of writing $t_{\tau,\underline{m},\underline{n}}$ for $\tau \in \mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$ in terms of the standard basis $B_{\mu,\nu}$. The Young symmetrizers y_{τ^+} and y_{τ^-} commute and have disjoint entries. Therefore, we can reduce our considerations to the case that ζ is a tableau of shape π for π a partition of some integer N, and here the Garnir relations can be applied.

We begin by focussing our attention on columns i and $i+1$ of ζ, which we assume are as displayed below:

$$\zeta = \begin{array}{cccc} & \underline{i} & \underline{i+1} & \\ \cdots & a_1 & b_1 & \cdots \\ & \vdots & \vdots & \\ \cdots & a_{s_1} & \vdots & \cdots \\ \cdots & \vdots & b_{t_1} & \cdots \\ & \vdots & \vdots & \\ \cdots & & b_{t_2} & \\ \cdots & a_{s_2} & & \\ \cdots & & & \end{array}$$

We further suppose that

(i) $X = \{a_{s_1}, \ldots, a_{s_2}\}$ is a subset of the set of entries $\{a_1, \ldots, a_{s_2}\}$ in the ith column of ζ;

(ii) $Y = \{b_1, \ldots, b_{t_1}\}$ is a subset of the set $\{b_1, \ldots, b_{t_2}\}$ of entries in the $(i+1)$st column of ζ for some $t_1 \geqslant s_1$;

(iii) $\{\rho_1, \ldots, \rho_k\}$ determines a complete set of coset representatives for $S_X \times S_Y$ in $S_{X \cup Y}$.

Then a *Garnir element* corresponding to X and Y is $g = \sum_{j=1}^{k} \operatorname{sgn}(\rho_j)\rho_j$.

LEMMA 3.8. *Suppose that $\pi \vdash N$ and let ζ be a tableau of shape π whose set of entries is $\mathscr{L}$.*

(i) *If γ is in the column group of ζ, then $y_\zeta = \operatorname{sgn}(\gamma)\gamma y_\zeta = \operatorname{sgn}(\gamma)\, y_{\gamma \cdot \zeta}\gamma$.*

(ii) *(Garnir relation) If g is a Garnir element, then $g \cdot y_\zeta = 0$. Thus, $y_\zeta = g' y_\zeta$, where $g' = -(g - \mathrm{id})$.*

Proof. Part (i) follows from the fact that for a permutation γ, $\gamma y_\zeta = \gamma y_\zeta \gamma^{-1}\gamma = y_{\gamma \cdot \zeta}\gamma$, while (ii) is well-known (see [P] or [JK], for example). ∎

Remark 3.9. Associated to ζ in $\mathscr{T}_\pi(\mathscr{L})$, the set of all tableaux of shape π with entries in $\mathscr{L}$, is the *Specht polynomial* $\mathfrak{f}(\zeta)$ obtained by replacing each entry j with the indeterminant x_j and then taking the product over all the differences of the entries in the columns. Thus, for

$$\zeta = \begin{array}{|c|c|c|}\hline 4 & 8 & 5 \\ \hline 3 & 1 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array} \qquad \text{we have} \qquad \begin{array}{|c|c|c|}\hline x_4 & x_8 & x_5 \\ \hline x_3 & x_1 & \\ \cline{1-2} x_6 & & \\ \cline{1-1} \end{array},$$

and $\mathfrak{f}(\zeta) = (x_4 - x_3)(x_4 - x_6)(x_3 - x_6)(x_8 - x_1)$. The $\mathbb{Z}$-span $\mathfrak{S}^\pi$ of all Specht polynomials corresponding to tableaux in $\mathscr{T}_\pi(\mathscr{L})$ is the *Specht module labeled by π for $S_{\mathscr{L}}$*, where the action is given by $\sigma\mathfrak{f}(\zeta) = \mathfrak{f}(\sigma \cdot \zeta)$. The polynomials corresponding to standard tableaux form a basis for $\mathfrak{S}^\pi$ [P, Theorem 1.1]. Thus, there is a unique expression $\mathfrak{f}(\zeta) = \sum_{i=1}^{l} \kappa_i \mathfrak{f}(\zeta_i) = \sum_{i=1}^{l} \kappa_i \mathfrak{f}(\omega_i \cdot \zeta) = (\sum_{i=1}^{l} \kappa_i \omega_i)\, \mathfrak{f}(\zeta)$, where $\omega_i \cdot \zeta =^{\mathrm{def}} \zeta_i \in \mathscr{S}\mathscr{T}_\pi(\mathscr{L})$ and $\omega_i \in S_{\mathscr{L}}$ for each i. Moreover, as Peel shows, the element $\mathfrak{g} =^{\mathrm{def}} \sum_{i=1}^{l} \kappa_i \omega_i \in \mathbb{Z}[S_{\mathscr{L}}]$ can be achieved by repeated applications of column permutations and Garnir relations. The argument in [P, Sect. 2] proves that the *Specht module* $\mathfrak{S}^\pi$ is isomorphic as a $S_{\mathscr{L}}$-module to the left ideal $\mathbb{Z}[S_{\mathscr{L}}]\, y_\zeta$ with $\mathfrak{f}(\sigma\zeta) = \sigma\mathfrak{f}(\zeta)$ mapping to $\sigma y_\zeta = y_{\sigma \cdot \zeta}\sigma$. Thus by the isomorphism, y_ζ has a unique expression: $y_\zeta = \sum_{i=1}^{l} \kappa_i \omega_i y_\zeta = \sum_{i=1}^{l} \kappa_i y_{\zeta_i} \omega_i$, where $\omega_i \cdot \zeta = \zeta_i \in \mathscr{S}\mathscr{T}_\pi(\mathscr{L})$ for each i and $\mathfrak{g} = \sum_i^l \kappa_i \omega_i \in \mathbb{Z}[S_{\mathscr{L}}]$ can be derived using column permutations and Garnir relations. (See also [JK, Sects. 7.2 and 8.1].)

Remark 3.10. If $\tau \in \mathscr{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$, then Remark 3.9 can be applied to $y_{\tau^*} \in \mathbb{C}[S_{\underline{m}^c}]$ and $y_{\tau_*} \in \mathbb{C}[S_{\underline{n}^c}]$ to produce a unique expression: $y_\tau = \sum_i^l \kappa_i y_{\tau_i} \omega_i$, where each rational tableau $\tau_i = \omega_i \cdot \tau$ is standard, $y_\tau = \mathfrak{g} y_\tau = \sum_i^l \kappa_i y_{\tau_i} \omega_i$, and $\mathfrak{g} = \sum_i^l \kappa_i \omega_i \in \mathbb{Z}[S_{\underline{m}^c} \times S_{\underline{n}^c}]$ can be obtained from column permutations and Garnir relations.

COROLLARY 3.11. *If* $\tau \in \mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$, *then there is a unique expression*:

$$t_{\tau,\underline{m},\underline{n}} = \sum_{i=1}^{l} \kappa_i t_{\tau_i,\underline{m},\underline{n}},$$

where $\tau_i = \omega_i \cdot \tau \in \mathcal{S}\mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$, *for each* i, $t_{\tau,\underline{m},\underline{n}} = \mathfrak{g} t_{\tau,\underline{m},\underline{n}} = \sum_i^l \kappa_i t_{\tau_i,\underline{m},\underline{n}}$, *and* $\mathfrak{g} = \sum_i^l \kappa_i \omega_i \in \mathbb{Z}[S_{\underline{m}^c} \times S_{\underline{n}^c}]$ *can be obtained from column permutations and Garnir relations.*

Proof. By Remark 3.10 we may assume that there exists a unique expression: $y_\tau = \sum_{i=1}^{l} \kappa_i y_{\tau_i} \omega_i$, where $\tau_i =^{\text{def}} \omega_i \cdot \tau \in \mathcal{S}\mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$ and $\kappa_i \in \mathbb{Z}$ for every i. Then since $\omega_i \in S_{\underline{m}^c} \times S_{\underline{n}^c}$ we have

$$\begin{aligned} t_{\tau,\underline{m},\underline{n}} = y_\tau c_{\underline{m},\underline{n}} \beta_{\tau,\underline{m},\underline{n}} &= \sum_{i=1}^{l} \kappa_i y_{\tau_i} \omega_i c_{\underline{m},\underline{n}} \beta_{\tau,\underline{m},\underline{n}} = \sum_{i=1}^{l} \kappa_i y_{\tau_i} c_{\underline{m},\underline{n}} \omega_i \beta_{\tau,\underline{m},\underline{n}} \\ &= \sum_{i=1}^{l} \kappa_i y_{\tau_i} c_{\underline{m},\underline{n}} \beta_{\tau_i,\underline{m},\underline{n}} = \sum_{i=1}^{l} \kappa_i t_{\tau_i,\underline{m},\underline{n}}. \quad \blacksquare \end{aligned}$$

EXAMPLE 3.12. Suppose $T = (\otimes^8 V) \otimes (\otimes^q V^*)$, where $q \geq 2$. Let $k = 2$ and suppose $(\underline{m}, \underline{n}) = \{(1, n_1), (3, n_2)\}$. Assume $\tau \in \mathcal{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$ and

$\tau^+ =$

7	4	8
5	2	
6		

which is nonstandard. For simplicity take τ^- to be standard. To rewrite $t_{\tau,\underline{m}\underline{n}}$ as a linear combination of standard basis elements perform the following steps:

(a) First rearrange the entries in the first two columns of $\zeta =^{\text{def}} \tau^+$ so that the resulting tableau has strictly increasing columns. Thus by (i), $y_\zeta = -(2\ 4)(5\ 6\ 7)\, y_\zeta = -y_{\zeta_1}(2\ 4)(5\ 6\ 7)$, where

$\zeta_1 = (2\ 4)(5\ 6\ 7) \cdot \zeta =$

5	2	8
6	4	
7		

.

(b) Apply (ii) to ζ_1 by taking $X = \{5, 6, 7\}$ and $Y = \{2\}$. A Garnir element corresponding to X and Y is given by $g = \text{id} - (2\ 5) + (2\ 6\ 5) - (2\ 7\ 6\ 5)$. Thus, $g y_{\zeta_1} = 0$, and it follows that

$$\begin{aligned} y_{\zeta_1} &= (2\ 5)\, y_{\zeta_1} - (2\ 6\ 5)\, y_{\zeta_1} + (2\ 7\ 6\ 5)\, y_{\zeta_1} \\ &= y_{\zeta_2}(2\ 5) - y_{\zeta_3}(2\ 6\ 5) + y_{\zeta_4}(2\ 7\ 6\ 5), \end{aligned}$$

where

$$\zeta_2 = (2\ \ 5)\cdot\zeta_1 = \begin{array}{|c|c|c|}\hline 2 & 5 & 8 \\ \hline 6 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} 7 & \multicolumn{2}{|c}{} \\ \cline{1-1}\end{array}\ ,$$

$$\zeta_3 = (2\ \ 6\ \ 5)\cdot\zeta_1 = \begin{array}{|c|c|c|}\hline 2 & 6 & 8 \\ \hline 5 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} 7 & \multicolumn{2}{|c}{} \\ \cline{1-1}\end{array}$$

$$\zeta_4 = (2\ \ 7\ \ 6\ \ 5)\cdot\zeta_1 = \begin{array}{|c|c|c|}\hline 2 & 7 & 8 \\ \hline 5 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} 6 & \multicolumn{2}{|c}{} \\ \cline{1-1}\end{array}\ .$$

(c) Correct the tableaux ζ_3 and ζ_4 by permuting the entries in the second column and applying (i). This gives $y_{\zeta_3} = -(4\ \ 6)\, y_{\zeta_3} = -y_{\zeta_5}(4\ \ 6)$ and $y_{\zeta_4} = -(4\ \ 7)\, y_{\zeta_4} = -y_{\zeta_6}(4\ \ 7)$, where

$$\zeta_5 = (4\ \ 6)\cdot\zeta_3 = \begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 5 & 6 & \multicolumn{1}{|c}{} \\ \cline{1-2} 7 & \multicolumn{2}{|c}{} \\ \cline{1-1}\end{array}\ ,$$

$$\zeta_6 = (4\ \ 7)\cdot\zeta_4 = \begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 5 & 7 & \multicolumn{1}{|c}{} \\ \cline{1-2} 6 & \multicolumn{2}{|c}{} \\ \cline{1-1}\end{array}\ .$$

(d) Standardize ζ_2 by first interchanging the entries in the second column using (i): $y_{\zeta_2} = -(4\ \ 5)\, y_{\zeta_2} = -y_{\zeta_7}(4\ \ 5)$, where

$$\zeta_7 = (4\ \ 5)\cdot\zeta_2 = \begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 6 & 5 & \multicolumn{1}{|c}{} \\ \cline{1-2} 7 & \multicolumn{2}{|c}{} \\ \cline{1-1}\end{array}\ .$$

(e) Next correct ζ_7 by applying the Garnir relations once again, this time with $X = \{6, 7\}$ and $Y = \{4, 5\}$ and

$$g = \mathrm{id} - (5\ \ 6) + (4\ \ 5\ \ 6) + (5\ \ 7\ \ 6) + (4\ \ 6)(5\ \ 7) - (4\ \ 5\ \ 7\ \ 6).$$

Use $gy_{\zeta_7} = 0$, to deduce

$$\begin{aligned} y_{\zeta_7} &= (5\ 6)\, y_{\zeta_7} - (4\ 5\ 6)\, y_{\zeta_7} - (5\ 7\ 6)\, y_{\zeta_7} - (4\ 6)(5\ 7)\, y_{\zeta_7} + (4\ 5\ 7\ 6)\, y_{\zeta_7} \\ &= y_{\zeta_8}(5\ 6) - y_{\zeta_9}(4\ 5\ 6) - y_{\zeta_{10}}(5\ 7\ 6) - y_{\zeta_{11}}(4\ 6)(5\ 7) + y_{\zeta_{12}}(4\ 5\ 7\ 6), \end{aligned}$$

where,

$$\zeta_8 = (5\ 6)\cdot\zeta_7 = \begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 5 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 7 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array},$$

$$\zeta_9 = (4\ 5\ 6)\cdot\zeta_7 = \begin{array}{|c|c|c|}\hline 2 & 5 & 8 \\ \hline 4 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 7 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

$$\zeta_{10} = (5\ 7\ 6)\cdot\zeta_7 = \begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array},$$

$$\zeta_{11} = (4\ 6)(5\ 7)\cdot\zeta_7 = \begin{array}{|c|c|c|}\hline 2 & 6 & 8 \\ \hline 4 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

and

$$\zeta_{12} = (4\ 5\ 7\ 6)\cdot\zeta_7 = \begin{array}{|c|c|c|}\hline 2 & 5 & 8 \\ \hline 4 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

Thus since $\zeta_5 = \zeta_8$ and $\zeta_6 = \zeta_{10}$, it follows that

$$\begin{aligned} y_{\tau^+} &= -y_{\zeta_1}(2\ 4)(5\ 6\ 7) \\ &= -y_{\zeta_2}(2\ 5)(2\ 4)(5\ 6\ 7) + y_{\zeta_3}(2\ 6\ 5)(2\ 4)(5\ 6\ 7) \\ &\quad - y_{\zeta_4}(2\ 7\ 6\ 5)(2\ 4)(5\ 6\ 7) \\ &= -y_{\zeta_2}(2\ 4\ 5\ 6\ 7) + y_{\zeta_3}(2\ 4\ 6\ 7) - y_{\zeta_4}(2\ 4\ 7) \\ &= y_{\zeta_7}(4\ 5)(2\ 4\ 5\ 6\ 7) - y_{\zeta_5}(4\ 6)(2\ 4\ 6\ 7) + y_{\zeta_6}(4\ 7)(2\ 4\ 7) \\ &= y_{\zeta_7}(2\ 5\ 6\ 7) - y_{\zeta_5}(2\ 6\ 7) + y_{\zeta_6}(2\ 7) \end{aligned}$$

$$= y_{\zeta_8}(5\ 6)(2\ 5\ 6\ 7) - y_{\zeta_9}(4\ 5\ 6)(2\ 5\ 6\ 7) - y_{\zeta_{10}}(5\ 7\ 6)(2\ 5\ 6\ 7)$$
$$- y_{\zeta_{11}}(4\ 6)(5\ 7)(2\ 5\ 6\ 7) + y_{\zeta_{12}}(4\ 5\ 7\ 6)(2\ 5\ 6\ 7)$$
$$- y_{\zeta_5}(2\ 6\ 7) + y_{\zeta_6}(2\ 7)$$
$$= - y_{\zeta_9}(2\ 6\ 7)(4\ 5) - y_{\zeta_{11}}(2\ 7)(4\ 6\ 5) + y_{\zeta_{12}}(2\ 7)(4\ 5).$$

Note that ζ_i for $i = 9, 11, 12$ is a standard tableau. Moreover, all the permutations involved in the above calculation belong to S_{m^c}, so they commute with $c_{m,n}$. Thus if $\tau_i = [\zeta_i, \tau^-]$ for $i = 9, 11, 12$, the expression for $t_{\tau,m,n}$ as a linear combination of elements of $B_{\mu,\nu}$ is given by $t_{\tau,m,n} = -t_{\tau_9,m,n} - t_{\tau_{11},m,n} + t_{\tau_{12},m,n}$.

4. The Irreducibility of the $\mathscr{A}$-Module $M_{\mu,\nu}$

In this section we prove that the $\mathscr{A}$-module $M_{\mu,\nu}$ for $[\mu, \nu] \in \Pi(p-k, q-k)$ is irreducible whenever $M_{\mu,\nu} \neq (0)$, and that $M_{\mu',\nu'}$ and $M_{\mu,\nu}$ are isomorphic as $\mathscr{A}$-modules if and only if $\mu' = \mu$ and $\nu' = \nu$. To prove irreducibility, we begin by introducing a bilinear form on T. The standard bases $\{v_1, ..., v_r\}$ of V and $\{v_1^*, ..., v_r^*\}$ of V^* are orthonormal relative to the bilinear forms $(v_i, v_j) = \delta_{i,j}$, and $(v_i^*, v_j^*) = \delta_{i,j}$. We extend these forms to obtain a bilinear form $\langle \cdot, \cdot \rangle$ on T defined by

$$\langle u_1 \otimes \cdots \otimes u_p \otimes u_{p+1}^* \otimes \cdots \otimes u_{p+q}^*, w_1 \otimes \cdots \otimes w_p \otimes w_{p+1}^* \otimes \cdots \otimes w_{p+q}^* \rangle$$
$$= (u_1, w_1)(u_2, w_2) \cdots (u_p, w_p)(u_{p+1}^*, w_{p+1}^*) \cdots (u_{p+q}^*, w_{p+q}^*).$$

The basis $\mathscr{B}$ of T of simple tensors is orthonormal relative to the form $\langle \cdot, \cdot \rangle$. If $t = \sum_{\beta \in \mathscr{B}} a_\beta \beta \in T$, we define $\bar{t} \in T$ by $\bar{t} = \sum_{\beta \in \mathscr{B}} \bar{a}_\beta \beta$, where $\bar{a}_\beta$ denotes the complex conjugate of a_β. Then for all nonzero $t \in T$, $\langle t, \bar{t} \rangle$ is a positive real number, and we have the following:

Lemma 4.1 (Compare [BBL, Lemma 2.19(ii)]. *For each* $1 \leqslant m \leqslant p$ *and* $p+1 \leqslant n \leqslant p+q$,

$$\langle c_{m,n} t, u \rangle = \langle t, c_{m,n} u \rangle$$

for all $t, u \in T$.

Theorem 4.2 (Compare [BBL, Theorem 2.20]). *Assume* $T_0 =^{\text{def}} T = (\otimes^p V) \otimes (\otimes^q V^*)$. *For* $1 \leqslant k \leqslant \min(p, q)$ *let*

$$T_k \stackrel{\text{def}}{=} \sum_{(m,n) \in P(k)} c_{m,n}(T).$$

When $k < \min(p, q)$ *suppose that* $\tilde{T}_k$ *is the subspace of all vectors in* T_k

which are annihilated by all $c_{\underline{m}',\underline{n}'}$, *where* $(\underline{m}',\underline{n}')\in P(k+1)$. *Then* $T_k=\tilde{T}_k\oplus T_{k+1}$ *for* $0\leqslant k\leqslant \min(p,q)-1$, *and*

$$T=\tilde{T}_0\oplus\tilde{T}_1\oplus\cdots\oplus\tilde{T}_{\min(p,q)-1}\oplus T_{\min(p,q)}$$

is an orthogonal decomposition of T into subspaces.

Proof. The mapping $t\to\bar{t}$ commutes with each contraction mapping, so $t\in T_k$ implies that $\bar{t}\in T_k$. Since $\langle t,\bar{t}\rangle>0$ for all nonzero t, the form is nondegenerate on each T_k. Now T_{k+1} is a subspace of T_k on which the form is nondegenerate. We claim that $(T_{k+1})^{\perp}=\tilde{T}_k$, where $(T_{k+1})^{\perp}$ is the orthogonal complement of T_{k+1} in T_k. Indeed, if $t\in\tilde{T}_k$ and $(\underline{m},\underline{n})\in P(k+1)$, then by Lemma 4.1 and Lemma 2.1(b)

$$\langle t, c_{\underline{m},\underline{n}}(u)\rangle=\langle c_{\underline{m},\underline{n}}(t),u\rangle=0$$

for all $u\in T$, to show that $t\in(T_{k+1})^{\perp}$. Conversely, if $t\in(T_{k+1})^{\perp}$ and $(\underline{m},\underline{n})\in P(k+1)$, then

$$0=\langle t,c_{\underline{m},\underline{n}}(u)\rangle=\langle c_{\underline{m},\underline{n}}(t),u\rangle,$$

for all u, and by the nondegeneracy of the form, $t\in\tilde{T}_k$. Hence $T_k=\tilde{T}_k\oplus T_{k+1}$, and we have the orthogonal decomposition

$$\begin{aligned}T&=\tilde{T}_0\oplus T_1=\tilde{T}_0\oplus\tilde{T}_1\oplus T_2=\cdots\\&=\tilde{T}_0\oplus\tilde{T}_1\oplus\cdots\oplus\tilde{T}_{\min(p,q)-1}\oplus T_{\min(p,q)}.\quad\blacksquare\end{aligned}$$

LEMMA 4.3. *Suppose* $[\mu,\nu]\in\Pi(p-k,q-k)$ *for some* $0\leqslant k\leqslant\min(p,q)$, *and assume* $r\geqslant l(\mu)+l(\nu)+k$. *Then* $M_{\mu,\nu}\subseteq T_k$, *and* $M_{\mu,\nu}\subseteq\tilde{T}_k$ *when* $k<\min(p,q)$.

Proof. If $t_{\tau,\underline{m},\underline{n}}\in B_{\mu,\nu}$, then $t_{\tau,\underline{m},\underline{n}}=c_{\underline{m},\underline{n}}\,y_\tau\beta_{\tau,\underline{m},\underline{n}}\in T_k$. Thus $M_{\mu,\nu}\subseteq T_k$ for all $k=0,1,\ldots,\min(p,q)$. Now assume that $k<\min(p,q)$. For each $c_{\underline{m}',\underline{n}'}\in P(k+1)$, $c_{\underline{m}',\underline{n}'}(M_{\mu,\nu})=(0)$ by Corollary 3.7. Therefore, $M_{\mu,\nu}\subseteq\tilde{T}_k$ as asserted. $\blacksquare$

COROLLARY 4.4. *Suppose* $[\mu,\nu]\in\Pi(p-k,q-k)$ *for some* $0\leqslant k\leqslant\min(p,q)$ *and assume* $r\geqslant l(\mu)+l(\nu)+k$. *If* $0\neq v\in M_{\mu,\nu}$, *then there exists* $(\underline{m},\underline{n})\in P(k)$ *so that* $c_{\underline{m},\underline{n}}v\neq 0$.

Proof. If $k=0$, then $c_{\underline{m},\underline{n}}=c_{\emptyset,\emptyset}=\text{id}$ suffices. Otherwise, assume $k>0$ and $v\in M_{\mu,\nu}$ with $c_{\underline{m},\underline{n}}v=0$ for all $(\underline{m},\underline{n})\in P(k)$. Then by Lemma 4.3, $v\in M_{\mu,\nu}\subseteq T_k\subseteq T_{k-1}$ and $M_{\mu,\nu}\subseteq\tilde{T}_k$ when $k<\min(p,q)$. Since by assumption $c_{\underline{m},\underline{n}}v=0$ for all $(\underline{m},\underline{n})\in P(k)$, and since $v\in T_{k-1}$, it follows that $v\in\tilde{T}_{k-1}$. Therefore, $v\in\tilde{T}_k\cap\tilde{T}_{k-1}=(0)$ when $k<\min(p,q)$, giving us the desired conclusion $v=0$ in that case. If $k=\min(p,q)$, then $v\in T_{\min(p,q)}\cap\tilde{T}_{\min(p,q)-1}=(0)$ to again show $v=0$. $\blacksquare$

THEOREM 4.5. *Let* $[\mu,\nu]\in\Pi(p-k,q-k)$ *for some* $0\leqslant k\leqslant\min(p,q)$.

Then $M_{\mu,\nu}$ is irreducible as an $\mathscr{A}$-module (and hence as a $\mathscr{C}$-module) for all $r \geqslant l(\mu)+l(\nu)+k$. In particular when $r \geqslant p+q$, $M_{\mu,\nu}$ is irreducible for all such pairs $[\mu, \nu]$ of partitions.

Proof. Suppose $0 \neq v \in M_{\mu,\nu}$. We argue that $\mathscr{A}v = M_{\mu,\nu}$. Since $B_{\mu,\nu}$ spans $M_{\mu,\nu}$, there is an expression for v,

$$v = \sum_{\substack{(\underline{m},\underline{n}) \in P(k) \\ \tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\underline{n}^c)}} a_{\tau,\underline{m},\underline{n}} t_{\tau,\underline{m},\underline{n}}, \tag{4.6}$$

where $a_{\tau,\underline{m},\underline{n}} \in \mathbb{C}$. By Corollary 4.4, there exists $(\underline{m}',\underline{n}') \in P(k)$ so that $c_{\underline{m}',\underline{n}'} v \neq 0$. From Corollary 3.6 we conclude that for each $(\underline{m},\underline{n}) \in P(k)$ either $c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} = 0$ or $c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} = r^s t_{\sigma\cdot\tau,\underline{m}',\underline{n}'}$ for some $s \in \mathbb{Z}^+ \cup \{0\}$ and some $\sigma \in S_{\mathscr{P}} \times S_{\mathscr{Q}}$ with $\sigma\cdot\tau \in \mathscr{T}_{\mu,\nu}((\underline{m}')^c, (\underline{n}')^c)$. The integer s and the permutation σ that arise depend on $(\underline{m},\underline{n})$, so we write $c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} = r^{s_{m,n}} t_{\sigma_{m,n}\cdot\tau,\underline{m}',\underline{n}'}$. Then

$$0 \neq c_{\underline{m}',\underline{n}'} v = \sum_{\substack{(\underline{m},\underline{n}) \in P(k) \\ \tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\underline{n}^c)}} \tilde{a}_{\tau,\underline{m},\underline{n}} t_{\sigma_{m,n}\cdot\tau,\underline{m}',\underline{n}'} \in \mathscr{A}v, \tag{4.7}$$

where

$$\tilde{a}_{\tau,\underline{m},\underline{n}} \stackrel{\text{def}}{=} \begin{cases} r^{s_{m,n}} a_{\tau,\underline{m},\underline{n}}, & \text{if } c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} = r^{s_{m,n}} t_{\sigma_{m,n}\cdot\tau,\underline{m}',\underline{n}'} \\ 0, & \text{if } c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m},\underline{n}} = 0. \end{cases}$$

The rational tableaux $\sigma_{\underline{m},\underline{n}}\cdot\tau$ in (4.7) are not necessarily standard. We therefore apply Corollary 3.11 to write each $t_{\sigma_{m,n}\cdot\tau,\underline{m}',\underline{n}'}$ as a $\mathbb{C}$-linear combination of elements in $B_{\mu,\nu}$. Then, after grouping coefficients, we obtain

$$0 \neq c_{\underline{m}',\underline{n}'} v = \sum_{\tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}((\underline{m}')^c,(\underline{n}')^c)} b_\tau t_{\tau,\underline{m}',\underline{n}'} \in \mathscr{A}v, \tag{4.8}$$

where $b_\tau \in \mathbb{C}$.

All of the rational tableaux in (4.8) are standard and have the same set of entries. Let τ' be minimal in the linear ordering of standard rational tableaux of shape $[\mu,\nu]$ such that $b_{\tau'} \neq 0$. Then by Lemma 2.3, $0 \neq y_{\tau'} c_{\underline{m}',\underline{n}'} v = b_{\tau'} t_{\tau',\underline{m}',\underline{n}'} \in \mathscr{A}v$. Since $S_{\mathscr{P}} \times S_{\mathscr{Q}}$ acts transitively on $B_{\mu,\nu}$ by Lemma 3.3, it follows that $B_{\mu,\nu} \subseteq \mathscr{A}v$. Thus $M_{\mu,\nu} = \mathscr{A}v$, to show that $M_{\mu,\nu}$ is irreducible. ∎

To prove that $M_{\mu',\nu'}$ and $M_{\mu,\nu}$ are nonisomorphic for distinct pairs $[\mu',\nu']$ and $[\mu,\nu]$ we make use of a partial ordering on pairs of partitions that is derived from the dominance ordering on partitions. Suppose that π and ϱ are partitions of N, and define a partial ordering by

$$\pi \preccurlyeq \varrho \Leftrightarrow \left(\sum_{k=1}^{i} \pi_k\right) \leqslant \left(\sum_{k=1}^{i} \varrho_k\right) \qquad \text{for all } i.$$

When $\pi \preccurlyeq \varrho$ we say that ϱ *dominates* π and call $\preccurlyeq$ the *dominance order*. The dominance order is stronger than the standard lexicographic order $\leqslant$. In fact, $\pi \preccurlyeq \varrho$ implies $\pi \leqslant \varrho$, but the other implication need not hold. For example, when $N = 6$, then $\{3^2\} < \{4, 1^2\}$ in the lexicographic order, but they are not comparable in the dominance order.

LEMMA 4.9 (Compare [JK]). *Suppose π and ϱ are partitions of N, and ζ_π and ζ_ϱ are tableaux of shape π and ϱ respectively having the same set of entries. If $(a\ b) \notin \mathscr{C}_{\zeta_\varrho}$ for every $(a\ b) \in \mathscr{R}_{\zeta_\pi}$, then $\pi \preccurlyeq \varrho$.*

COROLLARY 4.10. *Suppose π and ϱ are partitions of N, and ζ_π and ζ_ϱ are tableaux of shape π and ϱ respectively with the same set of entries. Then $y_{\zeta_\pi} y_{\zeta_\varrho} \neq 0$ implies $\pi \preccurlyeq \varrho$.*

Proof. Assume $y_{\zeta_\pi} y_{\zeta_\varrho} \neq 0$ and let $(a\ b) \in \mathscr{R}_{\zeta_\pi}$. If $(a\ b) \in \mathscr{C}_{\zeta_\varrho}$, then

$$y_{\zeta_\pi} y_{\zeta_\varrho} = (y_{\zeta_\pi}(a\ b))\, y_{\zeta_\varrho} = y_{\zeta_\pi}((a\ b)\, y_{\zeta_\varrho}) = -y_{\zeta_\pi} y_{\zeta_\varrho},$$

which contradicts $y_{\zeta_\pi} y_{\zeta_\varrho} \neq 0$. Therefore $(a\ b) \notin \mathscr{C}_{\zeta_\varrho}$, and by Lemma 4.9 $\pi \preccurlyeq \varrho$. ∎

We can extend the dominance partial order to pairs $[\mu', \nu']$ and $[\mu, \nu]$ of partitions having $|\mu'| = |\mu|$ and $|\nu'| = |\nu|$ by defining

$$[\mu', \nu'] \preccurlyeq [\mu, \nu] \Leftrightarrow \mu' \preccurlyeq \mu \text{ and } \nu' \preccurlyeq \nu.$$

LEMMA 4.11. *Suppose $[\mu', \nu']$ and $[\mu, \nu]$ are pairs of partitions in $\Pi(p-k, q-k)$ for some $k = 0, ..., \min(p, q)$. If $(\underline{m}, \underline{n}) \in P(k)$ and $\tau \in \mathscr{S}\mathscr{T}_{\mu',\nu'}(\underline{m}^c, \underline{n}^c)$, then*

$$[\mu', \nu'] \not\preccurlyeq [\mu, \nu] \qquad \textit{implies} \quad y_\tau c_{\underline{m},\underline{n}}(M_{\mu,\nu}) = 0.$$

Proof. Suppose that $[\mu', \nu'] \not\preccurlyeq [\mu, \nu]$ and there exists $(\underline{m}', \underline{n}') \in P(k)$ and $\tau' \in \mathscr{S}\mathscr{T}_{\mu,\nu}((\underline{m}')^c, (\underline{n}')^c)$, such that $y_\tau c_{\underline{m},\underline{n}} t_{\tau',\underline{m}',\underline{n}'} \neq 0$. Then we are in case (ii) of Corollary 3.6, and there exists $\sigma \in \bar{S}_{\mathscr{P}} \times \bar{S}_{\mathscr{Q}}$ such that $y_\tau c_{\underline{m},\underline{n}} t_{\tau',\underline{m}',\underline{n}'} = r^s y_\tau t_{\tau'',\underline{m},\underline{n}}$, where $s \in \mathbb{Z}^+ \cup \{0\}$ and $\tau'' = \sigma \cdot \tau' \in \mathscr{T}_{\mu,\nu}(\underline{m}^c, \underline{n}^c)$. Since $y_\tau t_{\tau'',\underline{m},\underline{n}} = y_\tau y_{\tau''} c_{\underline{m},\underline{n}} \beta_{\tau'',\underline{m},\underline{n}}$, we need to consider the product of Young symmetrizers, $y_\tau y_{\tau''} = y_{\tau^+} y_{\tau''^+} y_{\tau^-} y_{\tau''^-}$. If $y_{\tau^+} y_{\tau''^+} \neq 0$ and $y_{\tau^-} y_{\tau''^-} \neq 0$, then by Corollary 4.10, $[\mu', \nu'] \preccurlyeq [\mu, \nu]$, which is a contradiction. Thus, one of the products must be zero, and the lemma is proved. ∎

THEOREM 4.12. *Assume $r \geqslant \max(l(\mu) + l(\nu) + k, l(\mu') + l(\nu') + k')$ and suppose $[\mu', \nu'] \in \Pi(p-k', q-k')$ and $[\mu, \nu] \in \Pi(p-k, q-k)$ are pairs of partitions for $k, k' \in \{0, ..., \min(p, q)\}$. Then $M_{\mu',\nu'} \cong M_{\mu,\nu}$ as $\mathscr{A}$-modules if and only if $\mu' = \mu$ and $\nu' = \nu$.*

Proof. Assume that $M_{\mu',\nu'} \cong M_{\mu,\nu}$ and $[\mu',\nu'] \neq [\mu,\nu]$. Let $\phi: M_{\mu,\nu} \to M_{\mu',\nu'}$ be an $\mathscr{A}$-module isomorphism. Suppose first that $k' \neq k$. Without loss of generality, let $k < k'$, and let $(\underline{m}',\underline{n}') \in P(k')$. Then by Corollary 3.7, $c_{\underline{m}',\underline{n}'}(M_{\mu,\nu}) = (0)$, which implies

$$(0) = \phi(c_{\underline{m}',\underline{n}'}(M_{\mu,\nu})) = c_{\underline{m}',\underline{n}'}(\phi(M_{\mu,\nu})) = c_{\underline{m}',\underline{n}'}(M_{\mu',\nu'}).$$

However, if $\tau \in \mathscr{S}\mathscr{T}_{\mu',\nu'}((\underline{m}')^c,(\underline{n}')^c)$, then $t_{\tau,\underline{m}',\underline{n}'}$ is a standard basis element of $M_{\mu',\nu'}$ with $c_{\underline{m}',\underline{n}'} t_{\tau,\underline{m}',\underline{n}'} = r^k t_{\tau,\underline{m}',\underline{n}'} \neq 0$, to give a contradiction.

Thus, we can reduce to the case where $k' = k$. Since $[\mu',\nu'] \neq [\mu,\nu]$, either $[\mu',\nu'] \not\preccurlyeq [\mu,\nu]$ or $[\mu,\nu] \not\preccurlyeq [\mu',\nu']$, and without loss of generality we may assume $[\mu',\nu'] \not\preccurlyeq [\mu,\nu]$. Let $(\underline{m},\underline{n}) \in P(k)$ and let $\tau \in \mathscr{S}\mathscr{T}_{\mu',\nu'}(\underline{m}^c,\underline{n}^c)$. By Lemma 4.11, we have

$$0 = \phi(y_\tau c_{\underline{m},\underline{n}}(M_{\mu,\nu})) = y_\tau c_{\underline{m},\underline{n}}(\phi(M_{\mu,\nu})) = y_\tau c_{\underline{m},\underline{n}}(M_{\mu',\nu'}).$$

But $t_{\tau,\underline{m},\underline{n}}$ is a nonzero element of $M_{\mu',\nu'}$, and $y_\tau c_{\underline{m},\underline{n}} t_{\tau,\underline{m},\underline{n}} = r^k t_{\tau,\underline{m},\underline{n}} \neq 0$. Again we reach a contradiction, so we must have $\mu' = \mu$ and $\nu' = \nu$. ∎

5. The Centralizer Algebra

We prove that the algebra $\mathscr{A}$ generated by $S_{\mathscr{p}} \times S_{\mathscr{q}}$ and the contraction maps is the full centralizer algebra $\mathscr{C} = \operatorname{End}_{\mathscr{G}}(T)$ whenever $r \geqslant p+q$. Recently Koike [Koi] gave a proof of this fact based on the fundamental theorem of invariant theory. The argument we present here, which was done independently, is an application of the results in Section 4 on the modules $M_{\mu,\nu}$. We then identify $\mathscr{A} = \mathscr{C}$ with a subalgebra $\mathscr{B}_{p,q}^{(r)}$ of the Brauer algebra and give the decomposition of T as a $\mathscr{B}_{p,q}^{(r)} \times \mathscr{G}$-bimodule.

We assume that $[\mu,\nu] \in \Pi(p-k, q-k)$ for some $k = 0, ..., \min(p,q)$ and let $W_{\mu,\nu}$ be the $\mathscr{G}$-submodule of T spanned by the maximal vectors of weight $\lambda = \mu_1\varepsilon_1 + \cdots + \mu_{l(\mu)}\varepsilon_{l(\mu)} - \nu_{l(\nu)}\varepsilon_{r-l(\nu)+1} - \nu_1\varepsilon_r$, that is

$$W_{\mu,\nu} = \sum_{(\underline{m},\underline{n}) \in P(k)} \sum_{\tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\underline{n}^c)} \mathscr{U}(\mathscr{G}) t_{\tau,\underline{m},\underline{n}}.$$

Then $W_{\mu,\nu}$ is the $[\mu,\nu]$-*isotypic* component of T. Since $\mathscr{C}$ preserves weights and commutes with the action of $\mathscr{G}$, $W_{\mu,\nu}$ is a $\mathscr{C}$-module and hence an $\mathscr{A}$-module.

Assume that $r \geqslant p+q$, and let $m_{\mu,\nu} = (p!\,q!)/(k!\,h(\mu)\,h(\nu)) = \dim_{\mathbb{C}} M_{\mu,\nu}$ as in (2.13). Write $n_{\mu,\nu} \stackrel{\text{def}}{=} \dim_{\mathbb{C}} V([\mu,\nu]_r)$. Then as a $\mathscr{G}$-module,

$$W_{\mu,\nu} \cong \bigoplus \sum_{i=1}^{m_{\mu,\nu}} V([\mu,\nu]_r).$$

The $\mathbb{C}$-vector space $M_{\mu,\nu} \otimes V([\mu, \nu]_r)$ is a $\mathscr{C}$-module with action $c(t \otimes v) = ct \otimes v$ for all $c \in \mathscr{C}$, $t \in M_{\mu,\nu}$, and $v \in V([\mu, \nu]_r)$. It is a $\mathscr{U}(\mathscr{G})$-module where the action is given by $u(t \otimes v) = t \otimes uv$ for all $u \in \mathscr{U}(\mathscr{G})$, $t \in M_{\mu,\nu}$, and $v \in V([\mu, \nu]_r)$.

LEMMA 5.1. *As a* $\mathscr{C} \otimes \mathscr{U}(\mathscr{G})$*-module,* $W_{\mu,\nu} \cong M_{\mu,\nu} \otimes V([\mu, \nu]_r)$ *when* $r \geq p+q$.

Proof. Let v^+ be the unique (up to scalar multiple) maximal vector of $V([\mu, \nu]_r)$. Then for $u \in \mathscr{U}(\mathscr{G})$, the map $u \cdot t_{\tau,\underline{m},\underline{n}} \to u \cdot v^+$ is a $\mathscr{U}(\mathscr{G})$-module isomorphism between $\mathscr{U}(\mathscr{G}) \cdot t_{\tau,\underline{m},\underline{n}}$ and $V([\mu, \nu]_r)$. We can define a linear transformation ϕ: $W_{\mu,\nu} \to M_{\mu,\nu} \otimes V([\mu, \nu]_r)$ by

$$\phi\left(\sum_{(\underline{m},\underline{n}) \in P(k)} \sum_{\tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\underline{n}^c)} u_{\tau,\underline{m},\underline{n}} \cdot t_{\tau,\underline{m},\underline{n}}\right) = \sum_{(\underline{m},\underline{n}) \in P(k)} \sum_{\tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\underline{n}^c)} t_{\tau,\underline{m},\underline{n}} \otimes u_{\tau,\underline{m},\underline{n}} \cdot v^+,$$

where $u_{\tau,\underline{m},\underline{n}} \in \mathscr{U}(\mathscr{G})$. It is evident that ϕ respects both the $1 \otimes \mathscr{U}(\mathscr{G})$-action and $\mathscr{C} \otimes 1$-action and takes a basis onto a basis, so ϕ is a module isomorphism of the required type. ∎

COROLLARY 5.2. *When* $r \geq p+q$, *the tensor product* T *decomposes into irreducible* $\mathscr{A}$*-modules (and irreducible* $\mathscr{C}$*-modules) as follows,*

$$T \cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} n_{\mu,\nu} M_{\mu,\nu},$$

and thus $\mathscr{A}$ *and* $\mathscr{C}$ *are semisimple algebras.*

Proof. As a $\mathscr{C}$-module (and thus as an $\mathscr{A}$-module), $M_{\mu,\nu} \otimes V([\mu, \nu]_r) \cong \bigoplus \sum_{i=1}^{n_{\mu,\nu}} M_{\mu,\nu}$. Therefore,

$$\begin{aligned} T &= \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} W_{\mu,\nu} \\ &\cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} M_{\mu,\nu} \otimes V([\mu, \nu]_r) \\ &\cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} \left(\bigoplus \sum_{i=1}^{n_{\mu,\nu}} M_{\mu,\nu}\right) \\ &\cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} n_{\mu,\nu} M_{\mu,\nu}. \end{aligned}$$

Any element of the Jacobson radical $\operatorname{Rad}(\mathscr{C})$ or $\operatorname{Rad}(\mathscr{A})$ is zero on all irreducible modules and hence on T. As $\mathscr{C}$ and $\mathscr{A}$ are defined as algebras of transformations on T, they must then be semisimple. ▮

THEOREM 5.3. *If $r \geqslant p+q$, then $\mathscr{A} = \mathscr{C}$.*

Proof. The decomposition of T as a $\mathscr{U}(\mathscr{G})$-module into irreducible summands is given by

$$\begin{aligned} T &= \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} \left(\bigoplus \sum_{(\underline{m},\underline{n}) \in P(k)} \sum_{\tau \in \mathscr{S}\mathscr{T}_{\mu,\nu}(\underline{m}^c,\, \underline{n}^c)} \mathscr{U}(\mathscr{G})\, t_{\tau,\underline{m},\underline{n}} \right) \\ &\cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} m_{\mu,\nu} V([\mu,\nu]_r), \end{aligned}$$

where $m_{\mu,\nu} = (p!\,q!)/(k!\,h(\mu)\,h(\nu))$ is the multiplicity of $V([\mu,\nu]_r)$ in T as in Theorem 2.12. It follows from [R, (4.6)] that

$$C \cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} \mathfrak{M}_{m_{\mu,\nu}}(\mathbb{C}),$$

where $\mathfrak{M}_{m_{\mu,\nu}}(\mathbb{C})$ is the algebra of $m_{\mu,\nu} \times m_{\mu,\nu}$ matrices with entries in $\mathbb{C}$. Therefore since $\mathscr{A} \subseteq \mathscr{C}$,

$$\dim_{\mathbb{C}} \mathscr{A} \leqslant \dim_{\mathbb{C}} \mathscr{C} = \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} m_{\mu,\nu}^2.$$

On the other hand, $\mathscr{A}$ is semisimple and so by the Wedderburn–Artin theorem

$$\mathscr{A} \cong \bigoplus \sum_{\xi} \mathfrak{M}_{d_\xi}(\mathbb{C}),$$

where ξ runs over all isomorphism classes of irreducible $\mathscr{A}$-modules, and d_ξ is the dimension of the corresponding irreducible $\mathscr{A}$-module. We have produced a set of pairwise nonisomorphic irreducible $\mathscr{A}$-modules, namely the $M_{\mu,\nu}$ for $[\mu,\nu] \in \Pi(p-k,\, q-k)$. Hence,

$$\begin{aligned} \dim_{\mathbb{C}} \mathscr{A} &\geqslant \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} (\dim_{\mathbb{C}} M_{\mu,\nu})^2 \\ &= \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} m_{\mu,\nu}^2 = \dim_{\mathbb{C}} \mathscr{C}. \end{aligned}$$

Consequently, $\mathscr{A} = \mathscr{C}$ as claimed. ▮

COROLLARY 5.4. *When $r \geqslant p+q$, the set $\{M_{\mu,\nu} \mid [\mu,\nu] \in \Pi(p-k, q-k), k=0, ..., \min(p,q)\}$ is a complete set of representatives for the isomorphism classes of irreducible modules of $\mathscr{A} = \mathscr{C} = \mathrm{End}_{\mathscr{G}}(T)$, and*

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathscr{G}}(T) = \sum_{k=0}^{\min(p,q)} (p-k)!\,(q-k)! \binom{p}{k}^2 \binom{q}{k}^2 (k!)^2 = (p+q)! \tag{5.5}$$

Proof. Only the last statement needs to be verified, and for that we have

$$\begin{aligned}
\dim_{\mathbb{C}} \mathrm{End}_{\mathscr{G}}(T) &= \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,q-k)} (\dim_{\mathbb{C}} M_{\mu,\nu})^2 \\
&= \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,q-k)} \left(\frac{p!\,q!}{k!\,h(\mu)\,h(\nu)}\right)^2 \\
&= \sum_{k=0}^{\min(p,q)} \left(\sum_{\mu \vdash p-k} \left(\frac{(p-k)!}{h(\mu)}\right)^2\right) \\
&\quad \times \left(\sum_{\nu \vdash q-k} \left(\frac{(q-k)!}{h(\nu)}\right)^2\right) \binom{p}{k}^2 \binom{q}{k}^2 (k!)^2.
\end{aligned} \tag{5.6}$$

The irreducible modules for the symmetric group S_{p-k} on $p-k$ letters are labeled by the partitions μ of $p-k$ and have a basis indexed by the standard tableaux of shape μ. Thus, the dimension of the irreducible module labeled by μ is $(p-k)!/h(\mu)$. (See the discussion at the end of Section 2.) Since these are inequivalent modules, and the group algebra $\mathbb{C}[S_{p-k}]$ is semisimple, it follows that $(p-k)! = \sum_{\mu \vdash p-k} ((p-k)!/h(\mu))$. Substituting that expression into (5.6) gives the first equality of (5.5). Now for the second we observe that comparing the coefficients of x^p on both sides of $(1+x)^p (1+x)^q = (1+x)^{p+q}$ gives

$$\sum_{k=0}^{\min(p,q)} \binom{p}{p-k}\binom{q}{k} = \sum_{k=0}^{\min(p,q)} \binom{p}{k}\binom{q}{k} = \binom{p+q}{p},$$

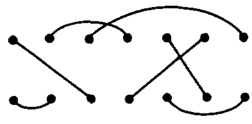
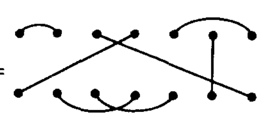
which implies

$$\begin{aligned}
\dim_{\mathbb{C}} \mathrm{End}_{\mathscr{G}}(T) &= \sum_{k=0}^{\min(p,q)} (p-k)!\,(q-k)! \binom{p}{k}^2 \binom{q}{k}^2 (k!)^2 \\
&= p!\,q! \sum_{k=0}^{\min(p,q)} \binom{p}{k}\binom{q}{k} = p!\,q! \binom{p+q}{p} = (p+q)! \quad \blacksquare
\end{aligned}$$

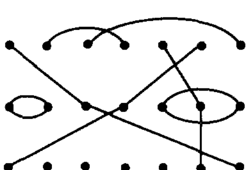
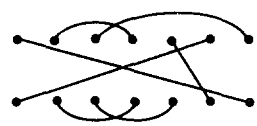
Let $O_r = O(r, \mathbb{C})$ be the orthogonal group, which we view as the group of isometries with respect to a symmetric, nondegenerate bilinear form $b(\cdot,\cdot)$ on $V = \mathbb{C}^r$. Then O_r acts naturally on $T^f = \bigotimes^f V$.

Assume $\{v_1^*, \ldots, v_r^*\} \subset V$ is the dual basis with respect to $b(\cdot, \cdot)$ so that $b(v_i, v_j^*) = \delta_{i,j}$. The action of $GL(r, \mathbb{C})$ on V^* when restricted to O_r is the same as the O_r-action on V if we identify v_i^* with v_i using the form. Therefore, $T = (\bigotimes^p V) \otimes (\bigotimes^q V^*) \cong T^{p+q}$ as O_r-representations. Since $\mathscr{C} = \mathrm{End}_{GL(r,\mathbb{C})}(T) \subseteq \mathrm{End}_{O_r}(T)$, it is natural to look for a copy of the centralizer algebra $\mathscr{C}$ inside of $\mathrm{End}_{O_r}(T)$.

In [Bra], Brauer showed that there is a homomorphism from the Brauer algebra $\mathscr{B}_f^{(r)}$ onto the centralizer algebra $\mathrm{End}_{O_r}(T^f)$ for any $f \geq 1$. To define the Brauer algebra, we say that an *f-diagram* is a graph on $2f$ vertices and f edges such that each vertex is incident to precisely one edge. We view f-diagrams as having their vertices arranged in 2 rows of f points, one above the other. For example,

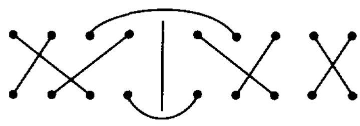
$d_1 =$  and $d_2 =$ 

are 7-diagrams. Let $t(d)$ and $b(d)$ denote respectively the top row and the bottom row of the f-diagram d. Edges joining vertices within each row are said to be *horizontal*, while edges joining vertices in the $t(d)$ with vertices in $b(d)$ are said to be *vertical*. For a parameter $x \in \mathbb{C}$, let $\mathscr{B}_f^{(x)}$ be the $\mathbb{C}$-vector space spanned by the f-diagrams. To describe multiplication in $\mathscr{B}_f^{(x)}$, let d_1 and d_2 be f-diagrams. Place d_1 above d_2 and identify vertices in $b(d_1)$ with the corresponding vertices in $t(d_2)$. The resulting graph consists of f paths whose endpoints are in $t(d_1) \cup b(d_2)$ along with a certain number γ of cycles which are adjacent to only vertices in the middle row. Let d be the f-diagram whose edges are the paths in this graph. Then the product of d_1 and d_2 is $d_1 d_2 = x^\gamma d$. For example, the product of the 7-diagrams shown above is

 $= x^2$  .

In general this product is not commutative, but $\mathscr{B}_f^{(x)}$ is an associative algebra whose identity element is the diagram having each vertex in the top row connected to the corresponding vertex in the bottom row. The papers [Wen, HW1, HW4] have studied $\mathscr{B}_f^{(x)}$ extensively.

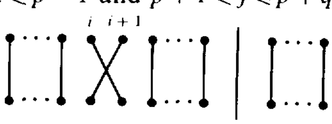
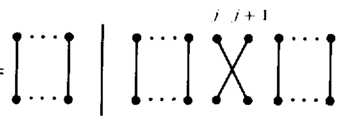
We say that a *(p, q)-diagram* is a $(p+q)$-diagram with a vertical wall between the pth and $(p+1)$st vertices of each row such that vertical edges never cross the wall and horizontal edges always begin and end on opposite sides of the wall. For example,



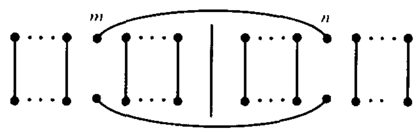
is a (4, 5)-diagram. We define $\mathscr{B}_{p,q}^{(x)}$ to be $\mathbb{C}$-span of the (p, q)-diagrams under the induced multiplication. It is not hard to see that $\mathscr{B}_{p,q}^{(x)}$ is closed under the product of (p, q)-diagrams and is thus a subalgebra of $\mathscr{B}_{p+q}^{(x)}$. The dimension of $\mathscr{B}_{p,q}^{(x)}$ is obtained by counting the diagrams with k horizontal edges in each row and then summing over k. Thus

$$(5.7)\quad \dim_{\mathbb{C}} \mathscr{B}_{p,q}^{(x)} = \sum_{k=0}^{\min(p,q)} \binom{p}{k}^2 \binom{q}{k}^2 (k!)^2 (p-k)!\,(q-k)! = (p+q)!,$$

where the last equality follows from (5.5). We number the vertices in each row of a (p, q)-diagram from left to right with $1, 2, \ldots, p+q$. For $1 \leqslant i \leqslant p-1$ and $p+1 \leqslant j \leqslant p+q-1$, let

$\sigma_i =$  and $\sigma_j^* =$ ,

and for $1 \leqslant m \leqslant p$ and $p+1 \leqslant n \leqslant p+q$, let

$d_{m,n} =$ .

One can check that these diagrams generate all of $\mathscr{B}_{p,q}^{(x)}$, and that the diagram with only vertical edges generate the subalgebra $\mathbb{C}[\mathscr{S}_p \times \mathscr{S}_q]$. In fact, each permutation $\sigma \in \mathscr{S}_p \times \mathscr{S}_q$ is identified with the diagram d_σ that has the ith vertex in $b(d_\sigma)$ connected to the $\sigma(i)$th vertex in $t(d_\sigma)$.

Using the invariant theory of O_r, Brauer [Bra] defined a representation $\phi\colon \mathscr{B}_f^{(r)} \to \operatorname{End}_{O_r}(T^f)$ of the Brauer algebra onto the centralizer O_r on T^f. The homomorphism can be described explicitly on f-diagrams. Let d be an f-diagram and define $\phi(d)$ to be the matrix whose $(\underline{i}, \underline{j})$ entry for $\underline{i} = i_1, \ldots, i_f$, and $\underline{j} = j_1, \ldots, j_f$ is determined by the following rules:

(1) Label the vertices in $t(d)$ from left to right with $v_{i_1}, v_{i_2}, \ldots, v_{i_f}$ and the vertices in $b(d)$ from left to right with $v_{j_1}^*, v_{j_2}^*, \ldots, v_{j_f}^*$.

(2) The $(\underline{i}, \underline{j})$-entry of $\phi(d)$ is the product of the values $b(u, w)$ over all the edges ε of d, where u and w are the labels on the vertices of ε.

If we restrict the representation $\phi: \mathscr{B}^{(r)}_{p+q} \to \mathrm{End}_{O_r}(T)$ to the subalgebra $\mathscr{B}^{(r)}_{p,q}$, we obtain a representation $\tilde{\phi}: \mathscr{B}^{(r)}_{p,q} \to \mathrm{End}_{O_r}(T)$. It is not hard to check that $\tilde{\phi}(d_\sigma) = \sigma \in \mathscr{S}_p \times \mathscr{S}_q$, where σ acts on T by place permutation, and that $\tilde{\phi}(d_{m,n}) = c_{m,n}$, where $c_{m,n}$ is the contraction map. Therefore $\tilde{\phi}(\mathscr{B}^{(r)}_{p,q}) = \mathscr{A} \subseteq \mathscr{C}$, and comparing dimensions we have:

THEOREM 5.8. *When* $r \geqslant p+q$, $\mathscr{B}^{(r)}_{p,q} \cong \mathscr{C} = \mathrm{End}_{GL(r,\mathbb{C})}(T)$.

Using the invariant theory of $GL(r, \mathbb{C})$, Koike [Koi] proves that $\mathscr{A} = \mathscr{C}$ for all positive integers r, and thus, we have that $\tilde{\phi}$ maps $\mathscr{B}^{(r)}_{p,q}$ onto $\mathscr{C}$ for all r and is an isomorphism when $r \geqslant p+q$.

Since the actions of $\mathscr{B}^{(r)}_{p,q}$ and $\mathscr{G}$ commute, the vector space T can be regarded as a bimodule for $\mathscr{B}^{(r)}_{p,q} \times \mathscr{G}$, where the action is given by $(b, x) \cdot t = bx \cdot t = xb \cdot t$, for $b \in \mathscr{B}^{(r)}_{p,q}$ and $x \in \mathscr{G}$. Using the proof of Corollary 5.2, i.e., Schur–Weyl duality, we obtain:

THEOREM 5.9. *If* $r \geqslant p+q$, *then as a bimodule for* $\mathscr{B}^{(r)}_{p,q} \times \mathscr{G}$,

$$T \cong \bigoplus \sum_{k=0}^{\min(p,q)} \sum_{[\mu,\nu] \in \Pi(p-k,\, q-k)} M_{\mu,\nu} \otimes V([\mu, \nu]_r),$$

where $M_{\mu,\nu}$ *is the irreducible* $\mathscr{B}^{(r)}_{p,q}$*-module labeled by the pair of partitions* $[\mu, \nu]$ *and* $V([\mu, \nu]_r)$ *is as in Theorem* 2.12.

Concluding Remarks. Weyl [Wey] showed that ϕ is an isomorphism when $[r/2] \geqslant p+q$. Later Brown [Bro1, Bro2] proved that ϕ is an isomorphism whenever $r \geqslant p+q$. This is a necessary restriction, for if $r = p+q-1$, the element $y = \sum_{\sigma \in \mathscr{S}_{p+q}} \mathrm{sgn}(\sigma)\sigma$ lies in the kernel of ϕ (and also of $\tilde{\phi}$). Indeed, when $r < p+q$, every simple tensor $v_{i_1} \otimes \cdots \otimes v_{i_{p+q}}$ has at least two subscripts that are the same. Then by the argument in Lemma 2.6, $y(v_{i_1} \otimes \cdots \otimes v_{i_{p+q}}) = 0$.

The fact that the expression in (5.5) equals $(p+q)!$ was pointed out to the authors by S. Okada after he received a copy of our manuscript. Using our algorithms $\mathfrak{A}$ and $\mathfrak{B}$ of Section 1, Okada has found a Robinson–Schensted type correspondence which affords a bijective proof of the fact that $\dim \mathscr{C} = (p+q)!$, a result which was also proved by Kosuda. Subsequently, Kosuda [Kos] has defined a quantum deformation $H^r_{m,n}(q)$ of $\mathscr{C}$ to describe the centralizer of mixed tensor representations of the quantum general linear group $\mathscr{U}_q(gl(r, \mathbb{C}))$. Halverson [Hal] has computed irreducible characters of and given branching rules for both $\mathscr{B}^{(r)}_{m,n}$ and $H^r_{m,n}(q)$ and has described $H^r_{m,n}(q)$ as a $\mathbb{C}(q)$-algebra defined on $(m, n; q)$-diagrams. Leduc [L] has constructed a 2-parameter $\mathbb{C}(z, q)$-algebra $\mathscr{A}_{m,n}(z, q)$ which is related to $H^r_{m,n}(q)$ and to $\mathscr{C}$ in the same way that the

Birman–Wenzl–Murakami algebra is related to the centralizer algebra of the quantum orthogonal group and to the Brauer algebra. Thus, when $z \to q^r$, $\mathscr{A}_{m,n}(z, q)$ becomes $H^r_{m,n}(q)$ and when $q \to 1$, $\mathscr{A}_{m,n}(z, q)$ becomes $\mathscr{B}^{(r)}_{m,n}$.

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