RSK Insertion for Diagram Algebras

and Multiplicity-Free Models

Tom Halverson
Macalester College
www.macalester/~halverson

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Joint Mathematics Meetings
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Some references:

- T Halverson and T Lewandowski, **RSK Insertion For Set Partitions and Diagram Algebras**, *Electronic J Combinatorics*, **11(2)** R24, 2004-2005.

- W Chen, E Deng, R Du, R Stanley, C Yan, **Crossings and Nestings of Matchings and Partitions**, *Trans. Amer. Math. Soc*., to appear.

- T Halverson, A Ram, **Partition Algebras**, *European J. Combin*., **26** (2005), 869--921.

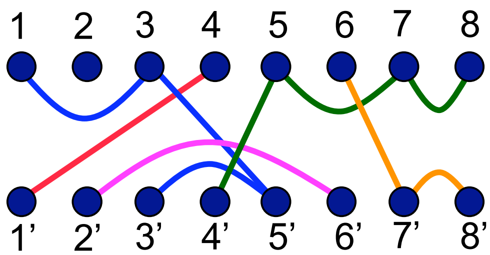
- Posters at this Meeting (Undergraduate Research Poster Session)
  - M Decker, RSK Insertion and a Model for the Symmetric Group
  - K Herbig, The Planar Rook Monoid

The Partition Algebra $P_k(n)$

P Martin 1991   V Jones 1993

(Potts Model in statistical mechanics)

Basis = partition diagrams on 2 rows of k vertices

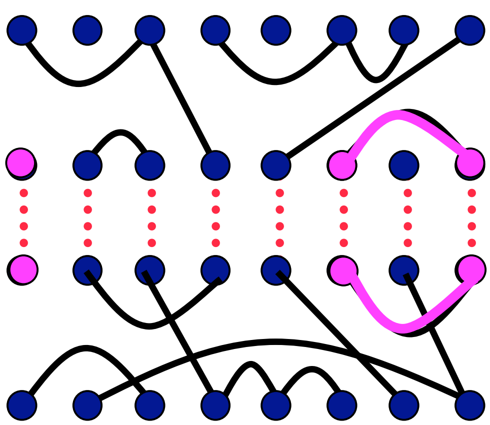


= { **{1,3,5',3'}, {1',4}, {2',6'}, {4',5,7,8}, {6,7',8'}, {2}** }

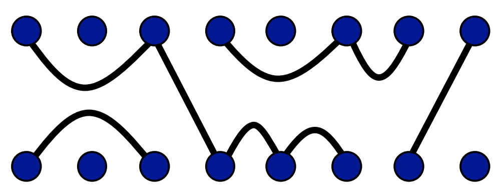
The dimension is the **Bell number B(2k)**

B(2k) = the number of partitions of {1, 2, …, 2k}

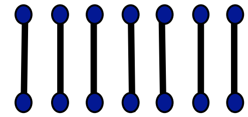
Multiplication $d_1 \, d_2 = n^f \, d_3$



$= n^2$



An associative algebra with identity $\mathbf{id} =$



Semisimple if $n \notin \{-2k, \ldots, 2k\}$

Schur-Weyl Duality S_n - $P_k(n)$

V = n-dimensional permutation representation of the symmetric group S_n

= $\text{span-}\{v_1, v_2, \ldots, v_n\}$, $\sigma v_i = v_{\sigma(i)}$

$V^{\otimes k}$ = k-fold tensor product representation

= $\text{span-}\left\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}\right\}$

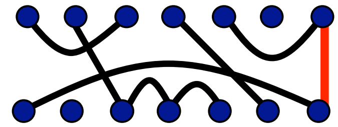
$P_k(n)$ = $\text{End}_{S_n}(V^{\otimes k})$ Centralizer algebra

$n \geq 2k$

As an S_n - $P_k(n)$ bimodule:

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k(n)} S_n^\lambda \otimes P_k^\lambda$$

Half-Integer Partition Algebras $P_{k+1/2}(n) \subseteq P_{k+1}(n)$



Tensor Identity:

$$\mathrm{Ind}_{S_{n-1}}^{S_n} \mathrm{Res}_{S_{n-1}}^{S_n} (V^{\otimes k}) \cong V^{\otimes k+1}$$

Partition Algebras:

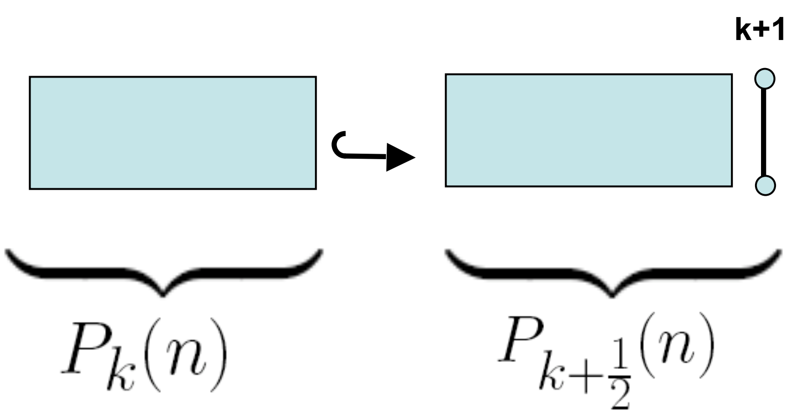
$$P_k(n) = \mathrm{End}_{S_n}(V^{\otimes k})$$ = span{ set partitions of {1, 2, …, 2k} }

$$P_{k+\frac{1}{2}}(n) = \mathrm{End}_{S_{n-1}}(V^{\otimes k})$$ = span{ set partitions of {1, 2, …, 2(k+1)}, (k+1) ~ 2(k+1) }

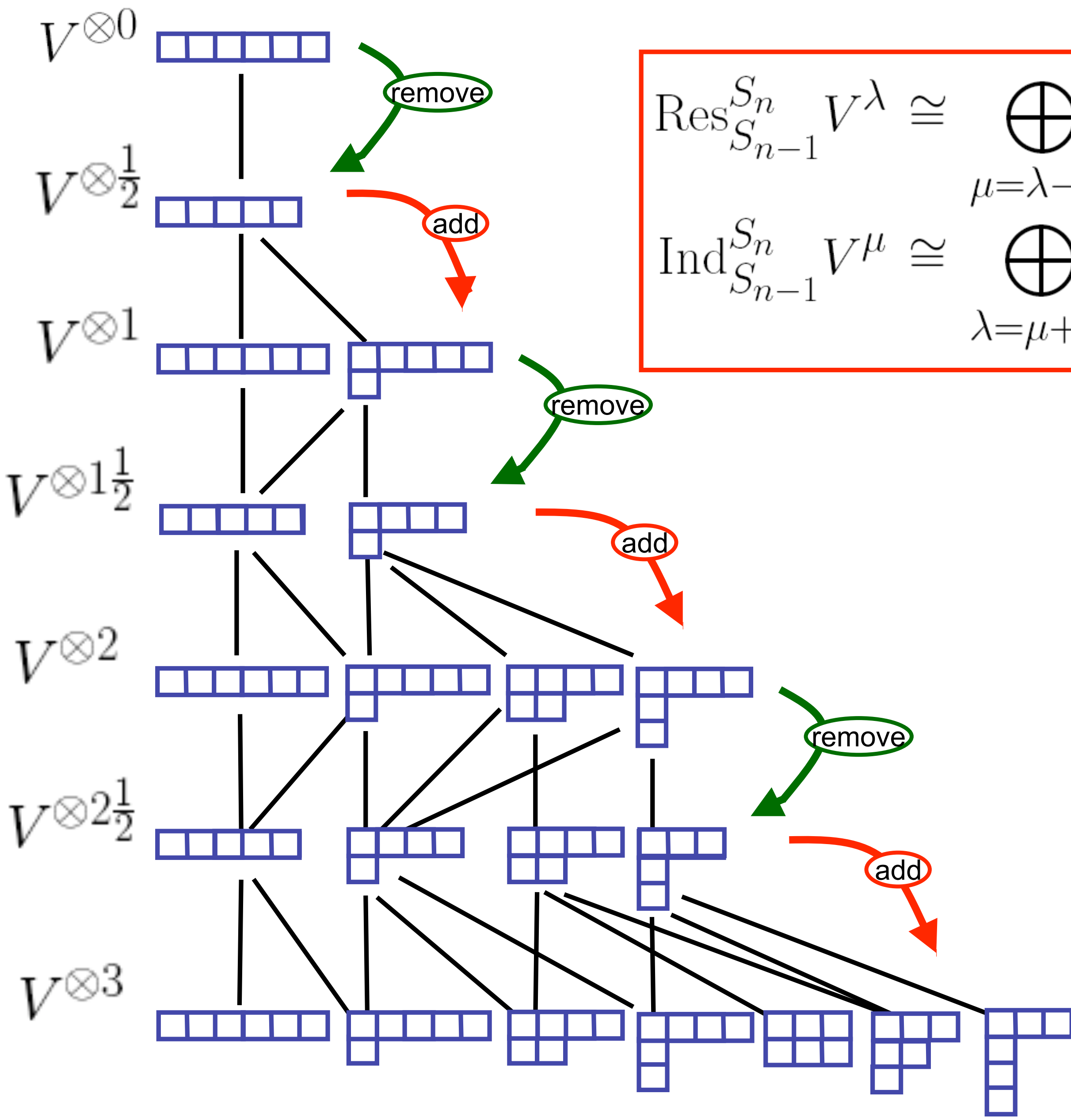
Dimension = B(2k+1), odd Bell numbers

Tower of Partition Algebras:

$$P_0(n) \subseteq P_{\frac{1}{2}}(n) \subseteq P_1(n) \subseteq P_{1\frac{1}{2}}(n) \subseteq P_2(n) \subseteq \cdots$$



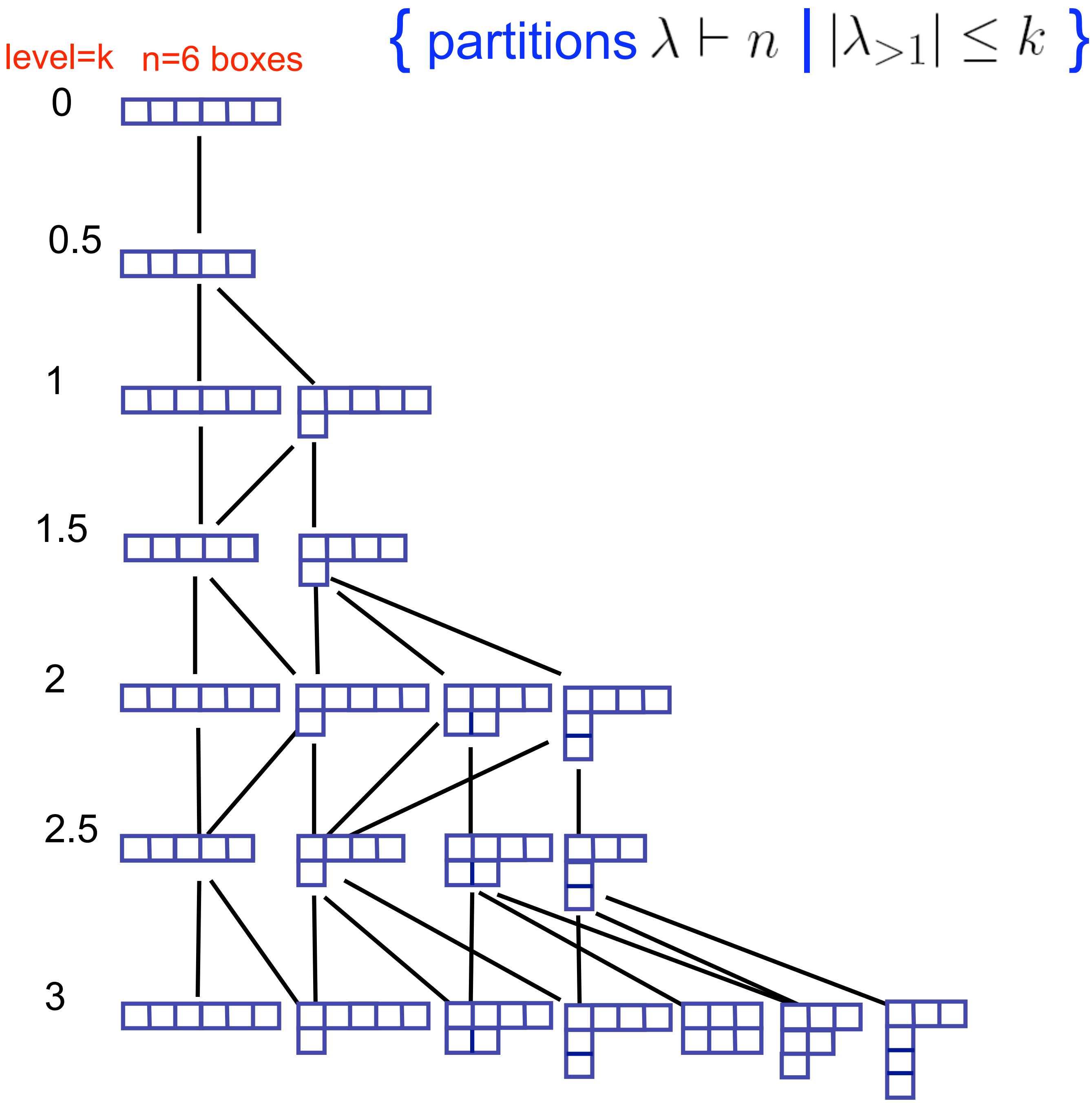
Restriction/Induction Rules for S_n:



$$\mathrm{Res}^{S_n}_{S_{n-1}} V^{\lambda} \cong \bigoplus_{\mu=\lambda-\square} V^{\mu}$$

$$\mathrm{Ind}^{S_n}_{S_{n-1}} V^{\mu} \cong \bigoplus_{\lambda=\mu+\square} V^{\lambda}$$

Alternate Labels

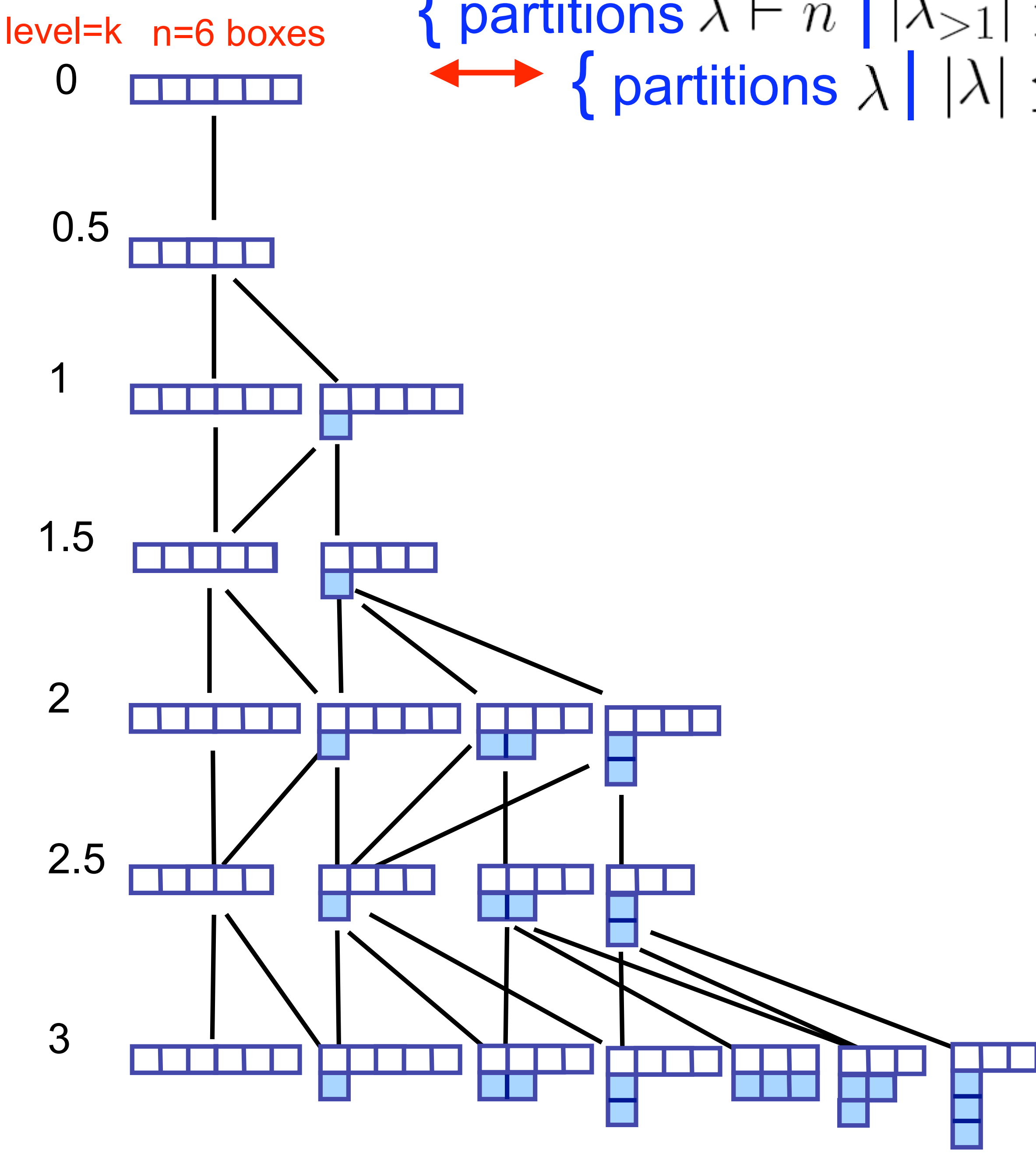


Alternate Labels

If n ≥ 2k then *ignore 1st row*

$\{$ partitions $\lambda \vdash n \mid |\lambda_{>1}| \leq k \}$

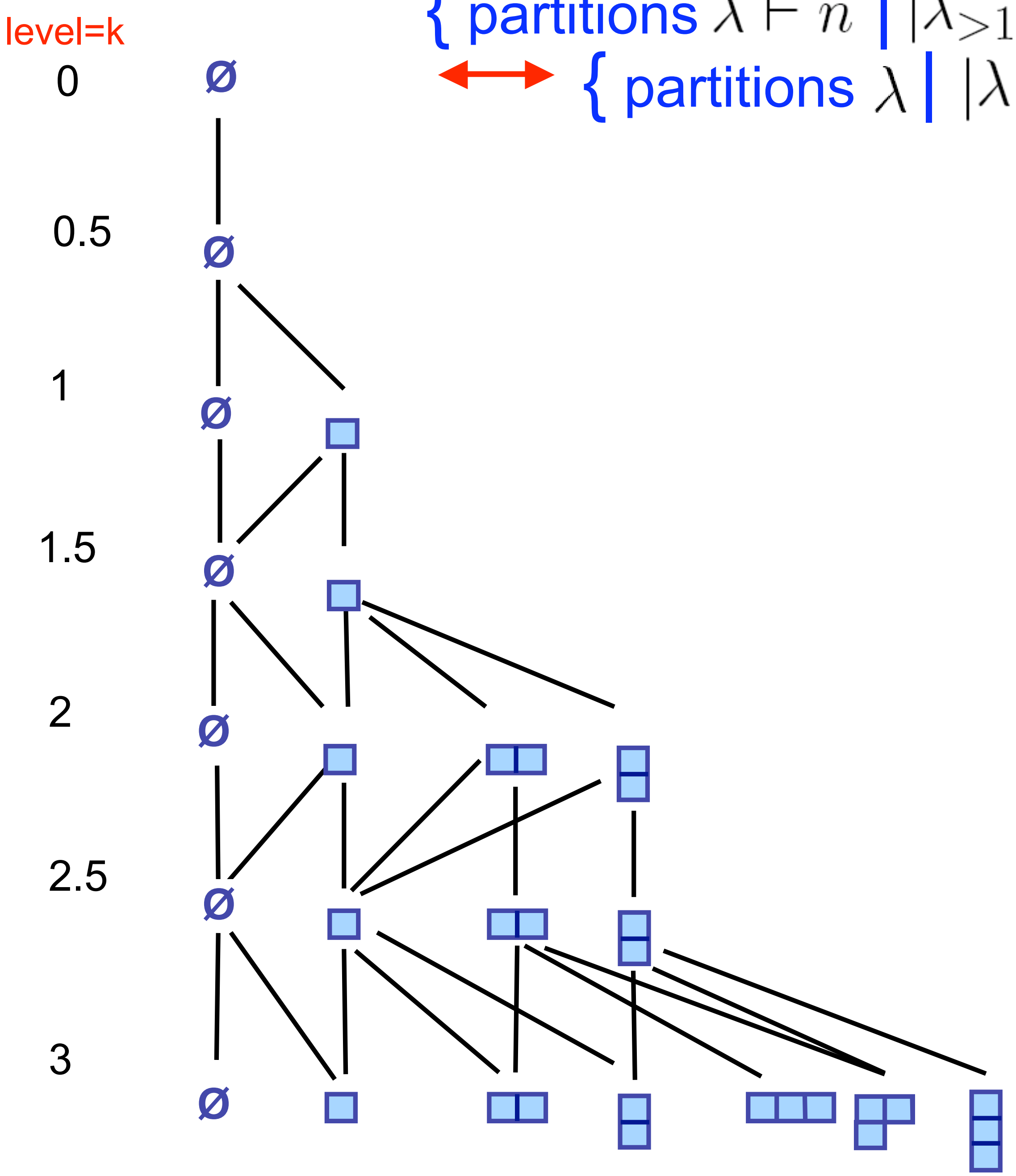
$\longleftrightarrow$ $\{$ partitions $\lambda \mid |\lambda| \leq k \}$



Alternate Labels

If n ≥ 2k then *ignore 1st row*

$\{$ partitions $\lambda \vdash n \mid |\lambda_{>1}| \leq k \}$ ⟷ $\{$ partitions $\lambda \mid |\lambda| \leq k \}$

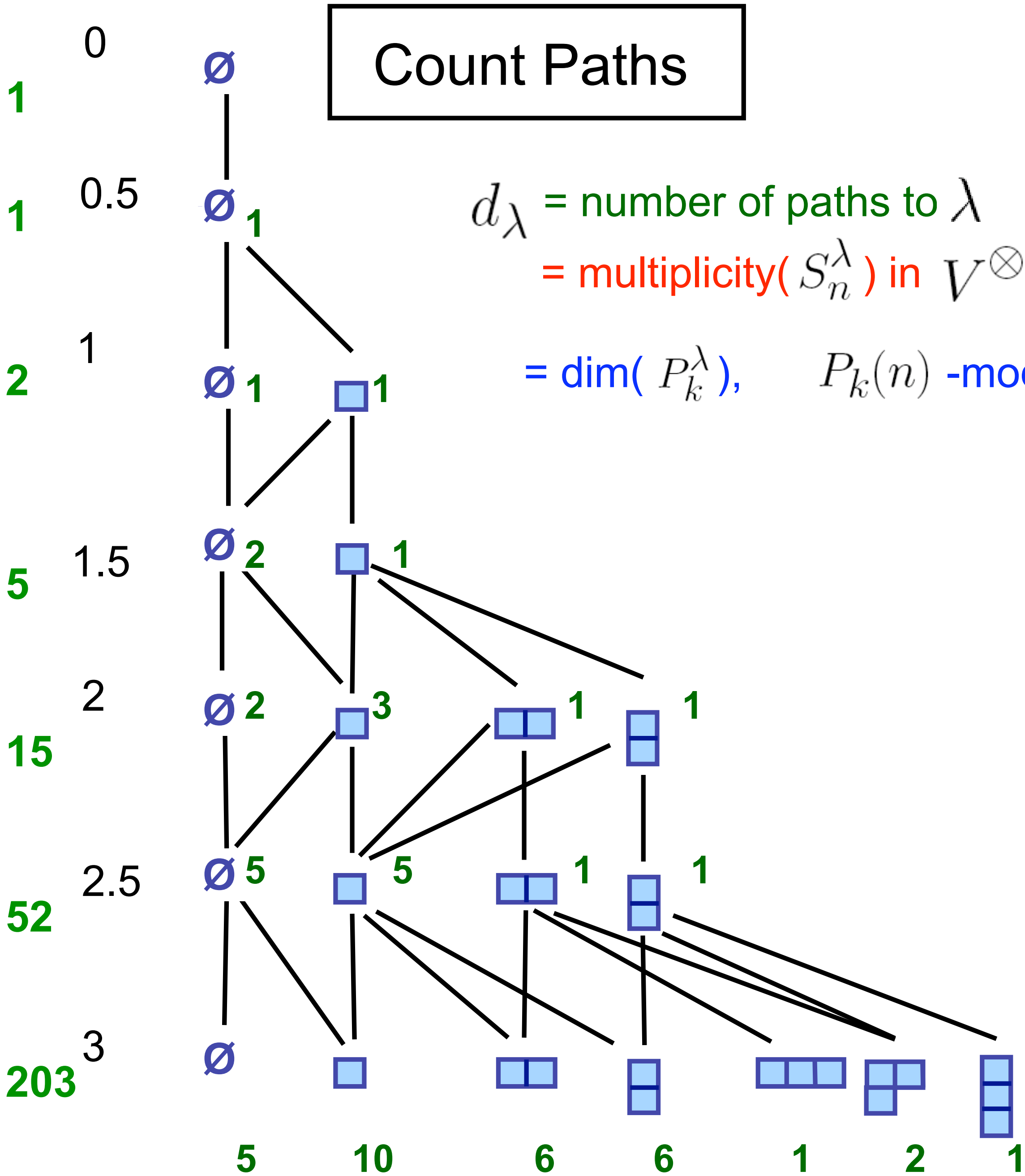


Count Paths

d_λ = number of paths to λ

= multiplicity(S_n^λ) in $V^{\otimes k}$

= dim(P_k^λ), $P_k(n)$ -module



$5^2 + 10^2 + 6^2 + 6^2 + 1^2 + 2^2 + 1^2 = 203 = B(6)$

$$\sum_\lambda d_\lambda^2 = B(2k)$$

RSK Bijections for the Partition Algebra

1. Regular Representation

$$P_k(n) \cong \bigoplus_{\substack{\lambda \\ |\lambda|\le k}} P_k^\lambda \otimes P_k^\lambda, \qquad \dim(P_k^\lambda) = d_\lambda$$

$$B(2k) = \sum_{\substack{\lambda \\ |\lambda|\le k}} d_\lambda^2$$

$$\{\text{partition diagrams}\} \leftrightarrow \{\text{pairs of paths } (P_\lambda, Q_\lambda)\}$$

2. Schur-Weyl Duality

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash n \\ |\lambda_{>1}|\le k}} S_n^\lambda \otimes P_k^\lambda$$

$$n^k = \sum_{\substack{\lambda \vdash n \\ |\lambda_{>1}|\le k}} f_\lambda \, d_\lambda$$

$$\{i_1, \ldots, i_k\} \leftrightarrow \{(F_\lambda, Q_\lambda)\}$$

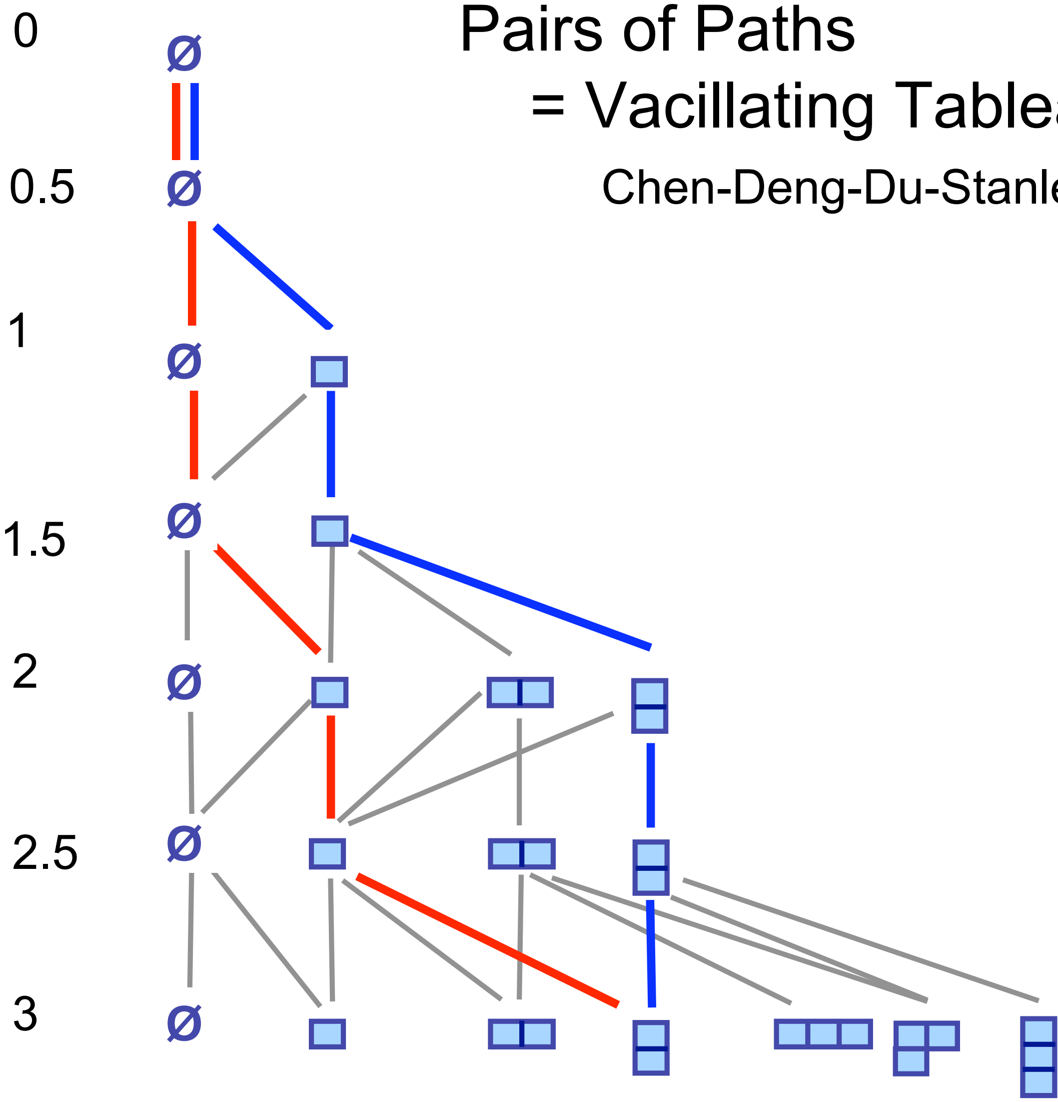
Standard Young tableau (F_λ)

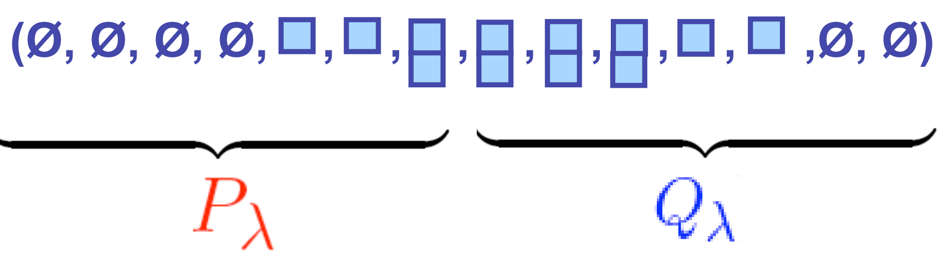
Path in $P_k(n)$ diagram (Q_λ)

Pairs of Paths

= Vacillating Tableaux

Chen-Deng-Du-Stanley-Yan





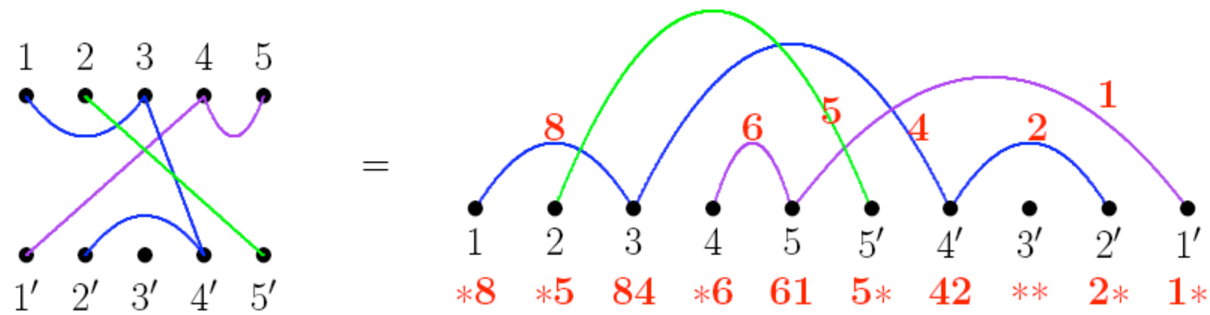
Vacillating tableaux:

- $\lambda^{i+\frac{1}{2}} = \lambda^i$ or $\lambda^i - \square$
- $\lambda^{i+1} = \lambda^{i+\frac{1}{2}}$ or $\lambda^{i+\frac{1}{2}} + \square$
- $\lambda^0 = \lambda^{2k} = \emptyset$

Insertion

Robinson, Schensted
Halverson-Lewandowski
Chen-Deng-Du-Stanley-Yan

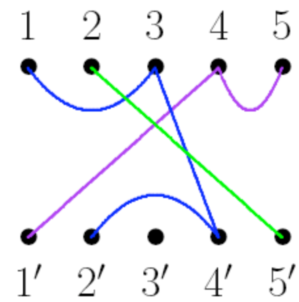
Sundaram, Roby, Stanley



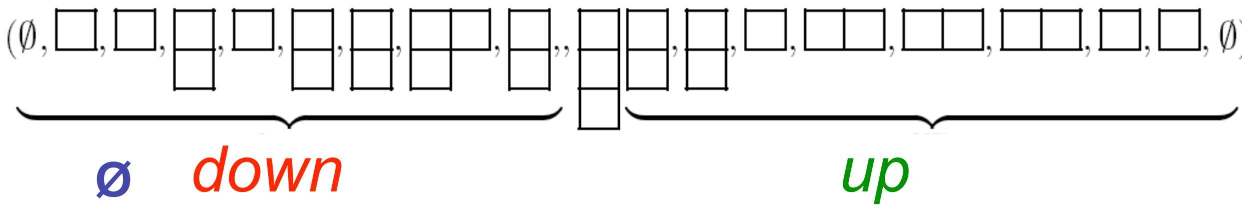
vertex	step	insertion	
1	$\frac{1}{2}$	∗	∅
	1	+8	8
2	$1\frac{1}{2}$	∗	8
	2	+5	5 8
3	$2\frac{1}{2}$	−8	5
	3	+4	4 5
4	$3\frac{1}{2}$	∗	4 5
	4	+6	4 6 5
5	$4\frac{1}{2}$	−6	4 5
	5	+1	1 4 5

vertex	step	insertion	
$1'$	$\frac{1}{2}'$	∗	∅
	$1'$	−1	1
$2'$	$1\frac{1}{2}'$	∗	1
	$2'$	−2	1 2
$3'$	$2\frac{1}{2}'$	∗	1 2
	$3'$	∗	1 2
$4'$	$3\frac{1}{2}'$	+2	1
	$4'$	−4	1 4
$5'$	$4\frac{1}{2}'$	∗	1 4
	$5'$	−5	1 4 5

Insertion



Remember only the shapes
get a vacillating tableau:



or, equivalently a pair of paths:

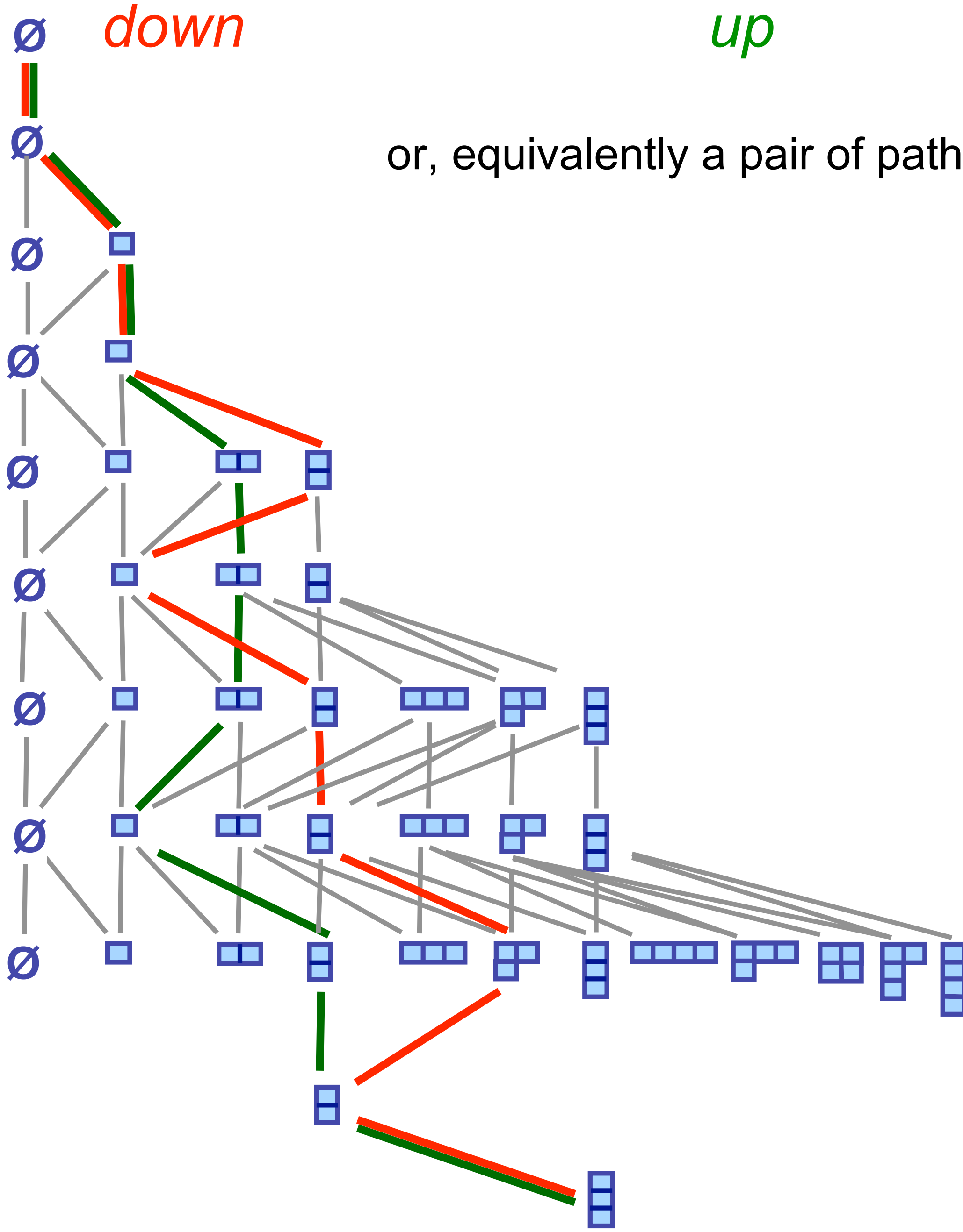
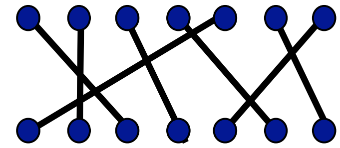


Diagram Subalgebras

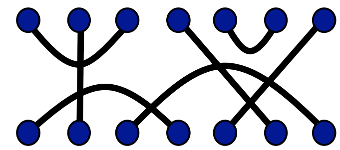
Symmetric Group Algebra

$$\mathbb{C}S_k = \text{span-}\left\{\begin{array}{l}\text{set partitions such that} \\ \text{each block consists of} \\ \textit{one top dot} \text{ and } \textit{one bottom dot}\end{array}\right.$$



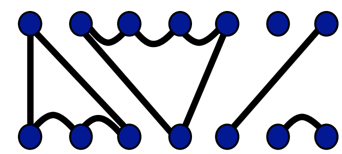
Brauer Algebra

$$B_k(n) = \text{span-}\left\{\begin{array}{c}\text{set partitions such that} \\ \text{each block consists of} \\ \textit{exactly two dots}\end{array}\right\}$$



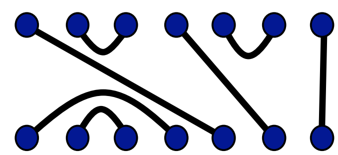
Planar Partition Algebra

$$PP_k(n) = \text{span-}\left\{\begin{array}{l}\text{set partitions that can} \\ \text{be represented} \\ \textit{without edge crossings}\end{array}\right\}$$



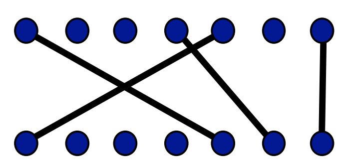
Temperley-Lieb Algebra

$$TL_k(n) = \text{span-}\left\{\begin{array}{c}\textit{planar} \text{ set partitions such} \\ \text{that each block consists of} \\ \textit{exactly two dots}\end{array}\right\}$$



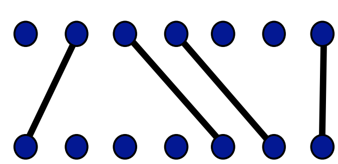
Rook Monoid Algebra

$$R_k(n) = \text{span-}\left\{\begin{array}{l}\text{each block is either} \\ \text{a } \textit{single dot} \quad \text{or} \\ \textit{one dot in each row}\end{array}\right\}$$



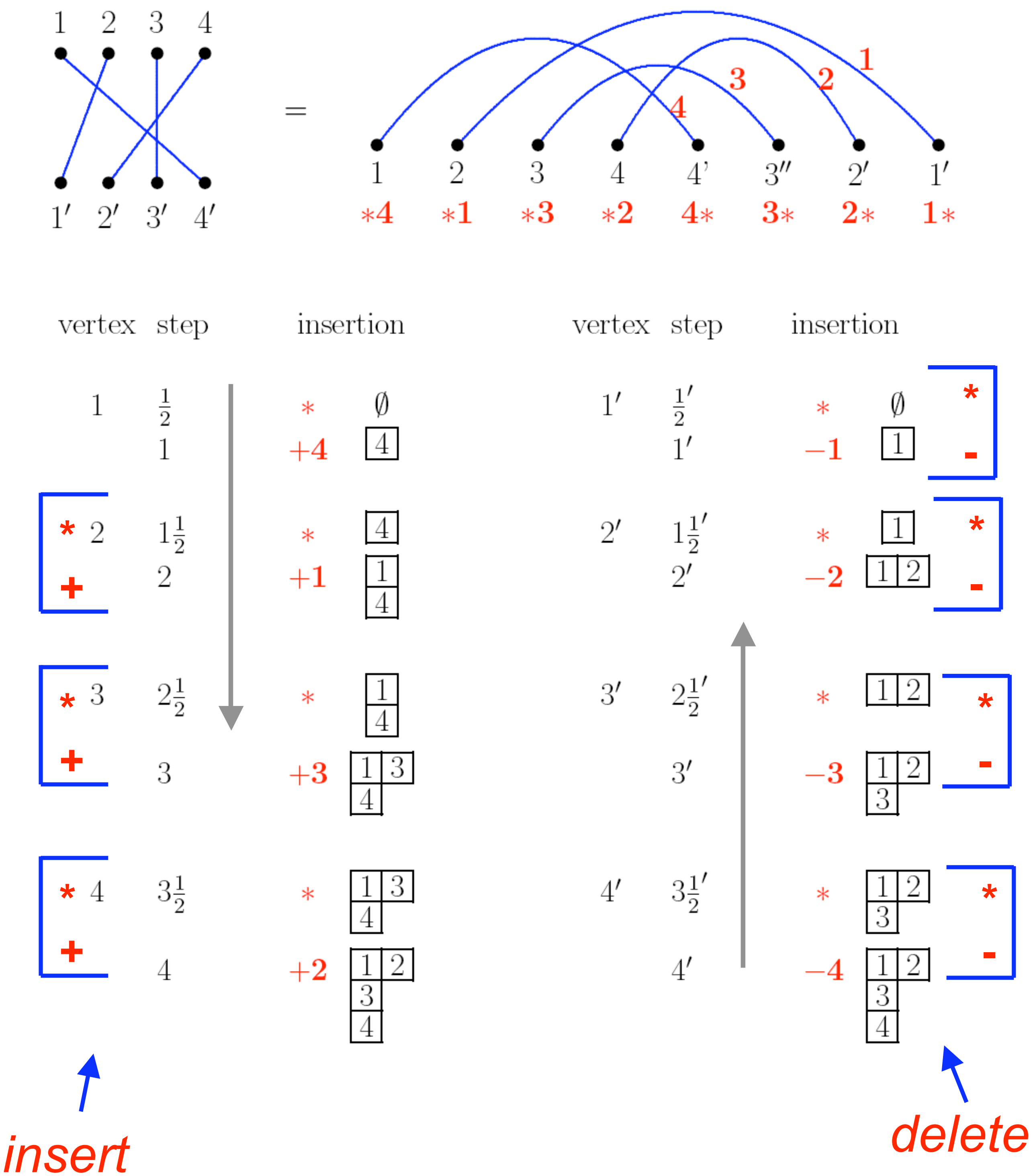
Planar Rook Monoid Algebra

$$PR_k(n) = \text{span-}\left\{\begin{array}{l}\textit{planar} \\ \text{rook monoid} \\ \text{diagrams}\end{array}\right\}$$

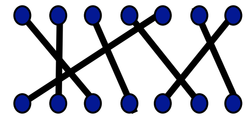


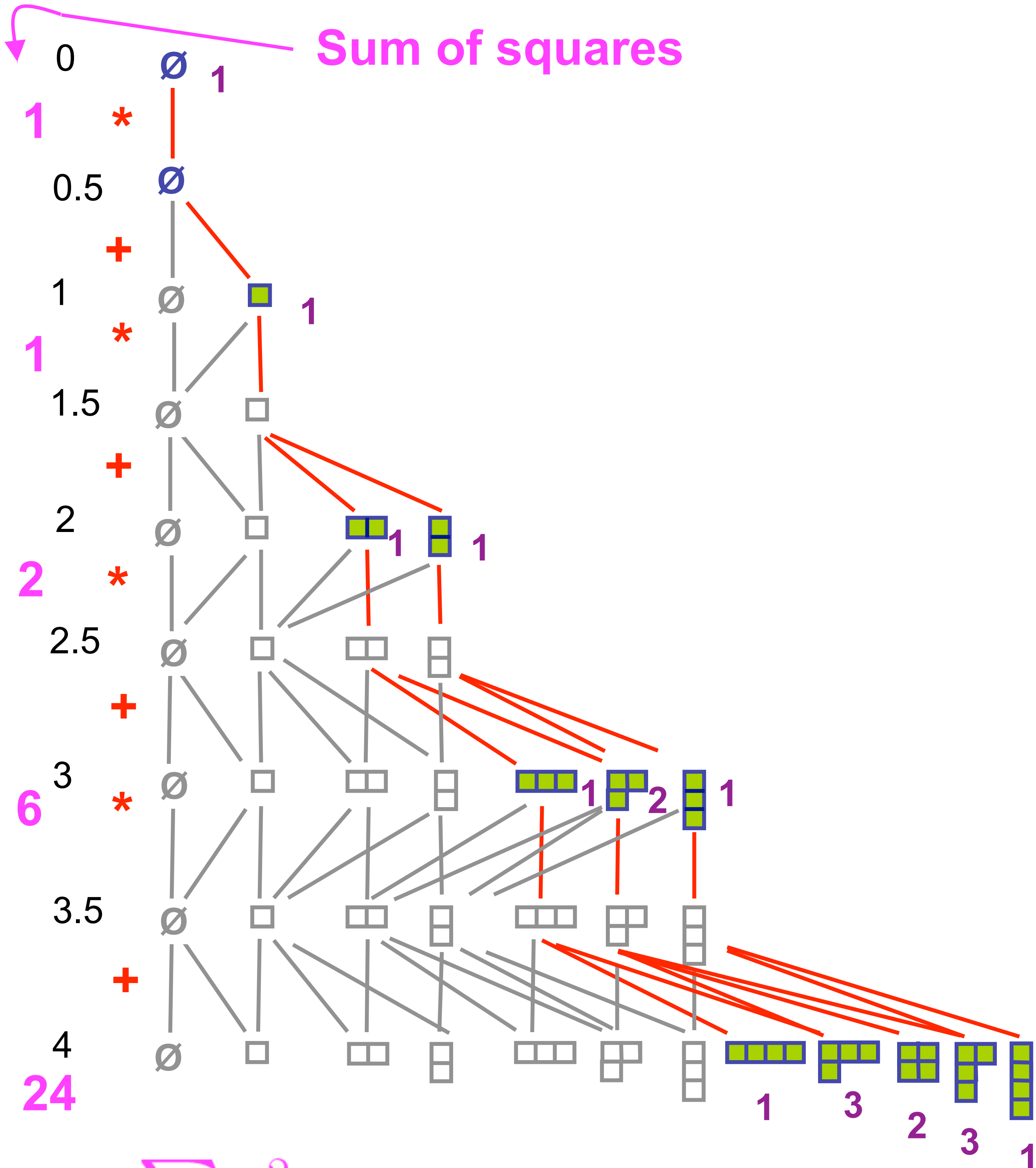
Walled Brauer Aglebra, Tanabe Algebras, …

Symmetric Group Diagrams



All Symmetric Group Diagrams



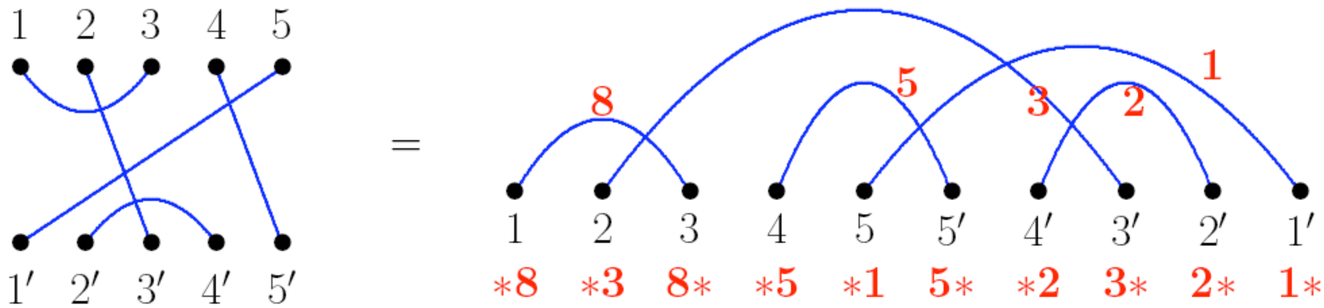


$$n! = \sum_{\lambda \vdash n} f_\lambda^2$$

(Standard tableaux)

Brauer Diagrams

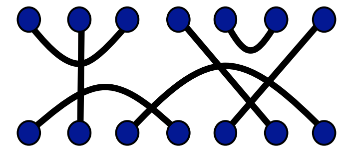
R. Brauer, 1937
S. Sundaram, 1986

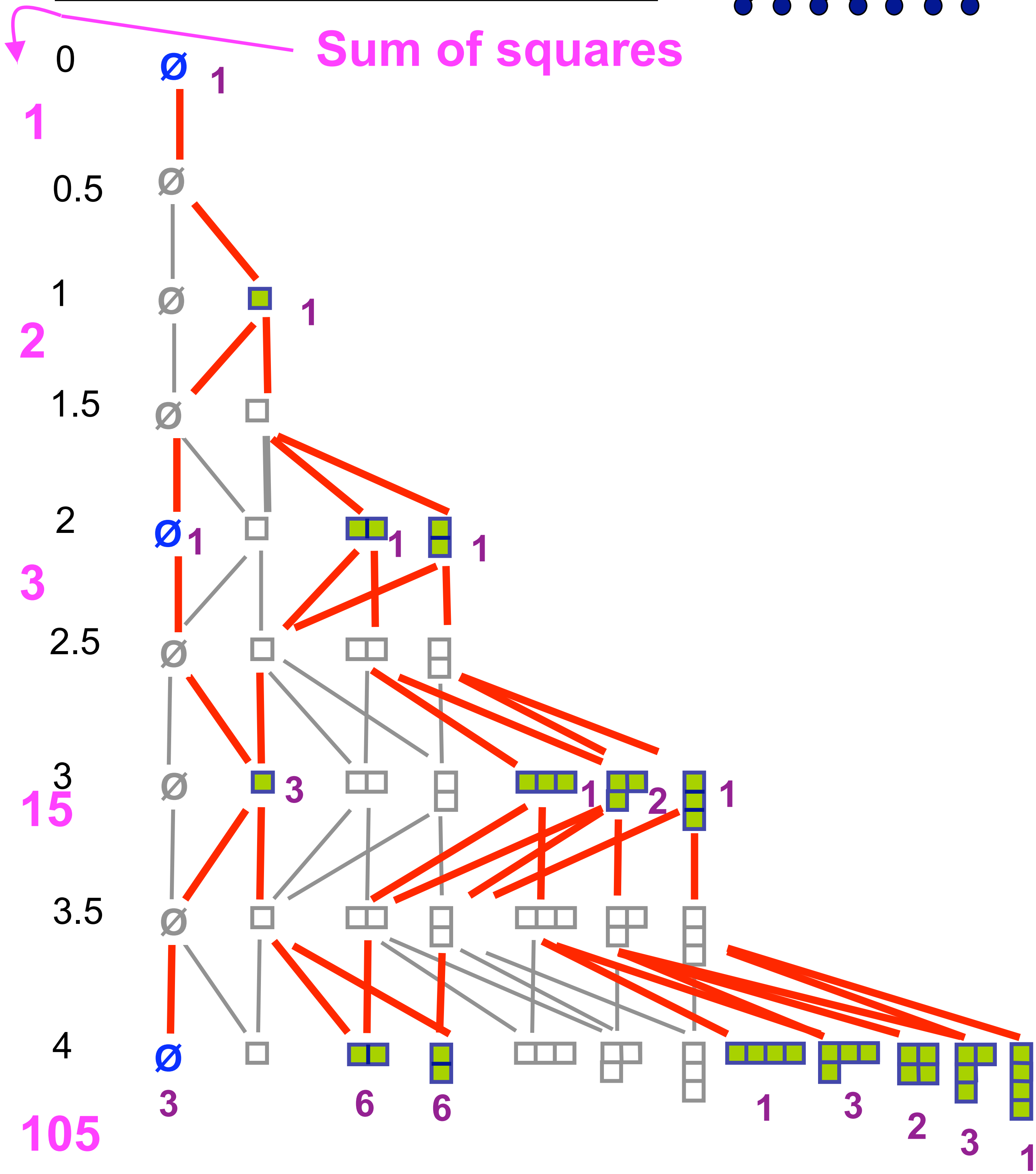


	vertex	step	insertion	
*	1	$\frac{1}{2}$	*	$\emptyset$
+		1	+8	8
*	2	$1\frac{1}{2}$	*	8
+		2	+3	3 / 8
-	3	$2\frac{1}{2}$	−8	3
*		3	*	3
*	4	$3\frac{1}{2}$	*	3
+		4	+5	3 5
*	5	$4\frac{1}{2}$	*	3 5
+		5	+1	1 5 / 3

vertex	step	insertion		
1′	$\frac{1'}{2}$	*	$\emptyset$	*
	1′	−1	1	-
2′	$1\frac{1'}{2}$	*	1	*
	2′	−2	1 2	-
3′	$2\frac{1'}{2}$	*	1 2	*
	3′	−3	1 2 / 3	-
4′	$3\frac{1'}{2}$	+2	1 / 3	+
	4′	*	1 / 3	*
5′	$4\frac{1'}{2}$	*	1 / 3	*
	5′	−5	1 5 / 3	-

Insert all Brauer diagrams



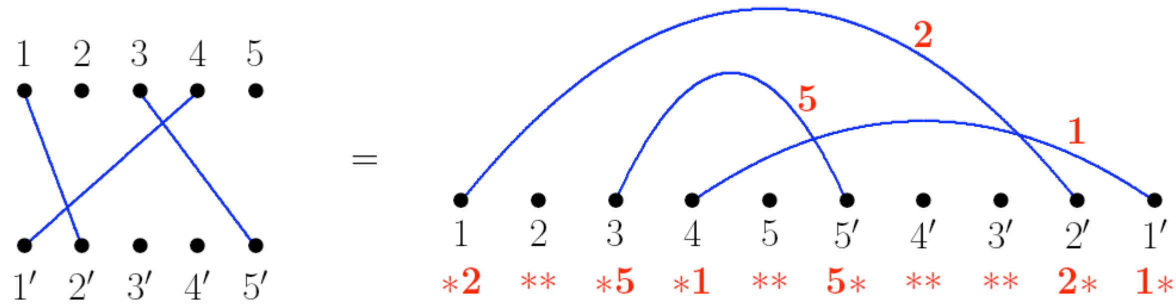


$(2k-1)!! = (2k-1)(2k-3)\cdots 3\cdot 1$

(Oscillating tableaux)

Rook Monoid Diagrams

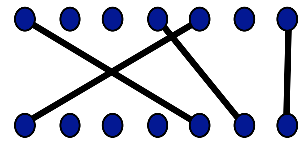
L. Solomon
C. Grood
L. Renner
Halverson

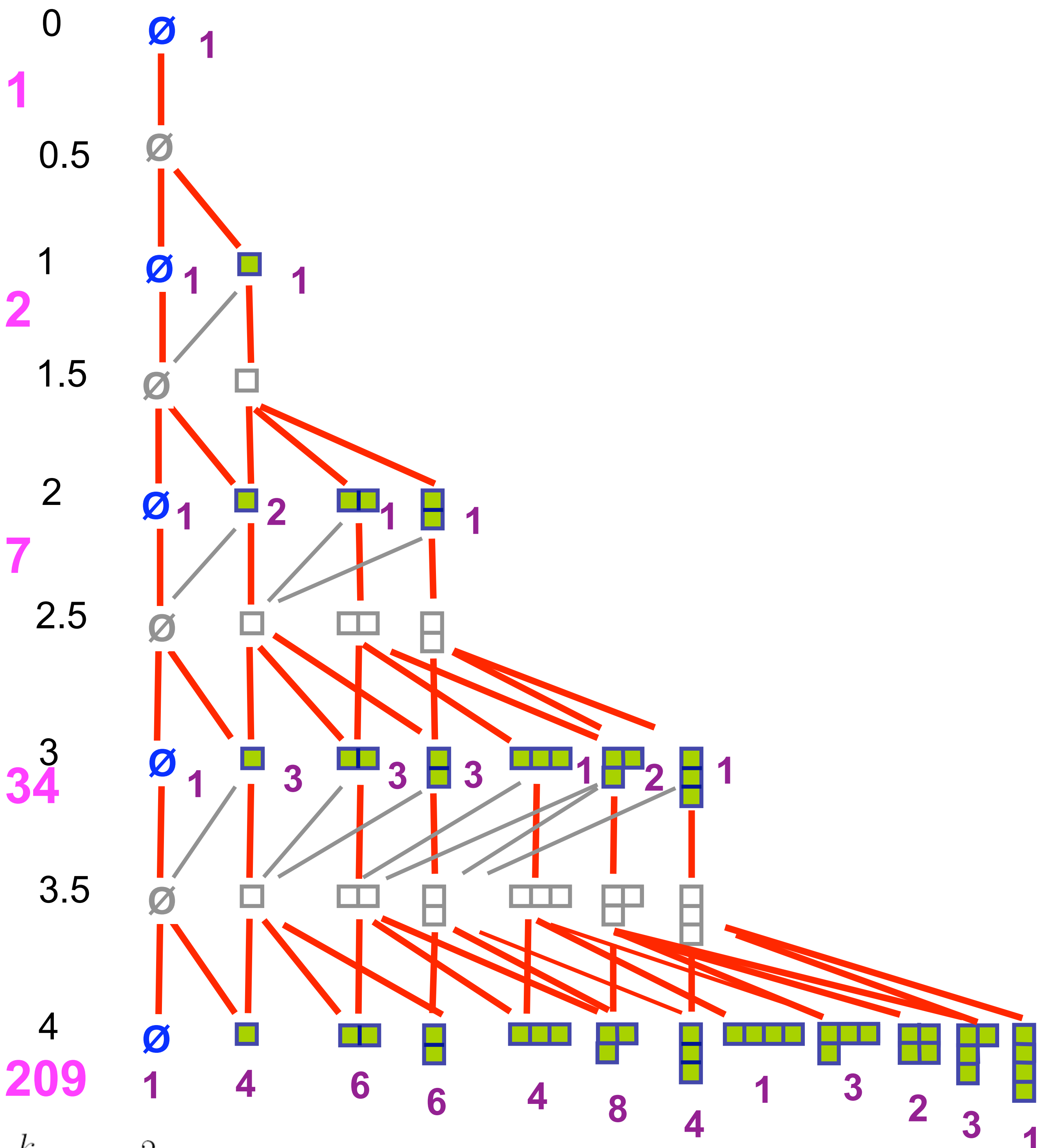


	vertex	step		insertion
*	1	$\frac{1}{2}$	*	∅
+		1	+2	2
*	2	$1\frac{1}{2}$	*	2
*		2	*	2
*	3	$2\frac{1}{2}$	*	2
+		3	+5	2 5
*	4	$3\frac{1}{2}$	*	2 5
+		4	+1	1 5 / 2
*	5	$4\frac{1}{2}$	*	1 5 / 2
*		5	*	1 5 / 3

vertex	step		insertion	
1′	$\frac{1}{2}'$	*	∅	*
	1′	−1	1	-
2′	$1\frac{1}{2}'$	*	1	*
	2′	−2	1 / 2	-
3′	$2\frac{1}{2}'$	*	1 / 2	*
	3′	*	1 / 2	*
4′	$3\frac{1}{2}'$	*	1 / 2	*
	4′	*	1 / 2	*
5′	$4\frac{1}{2}'$	*	1 / 2	*
	5′	−5	1 5 / 2	-

Insert all rook monoid diagrams

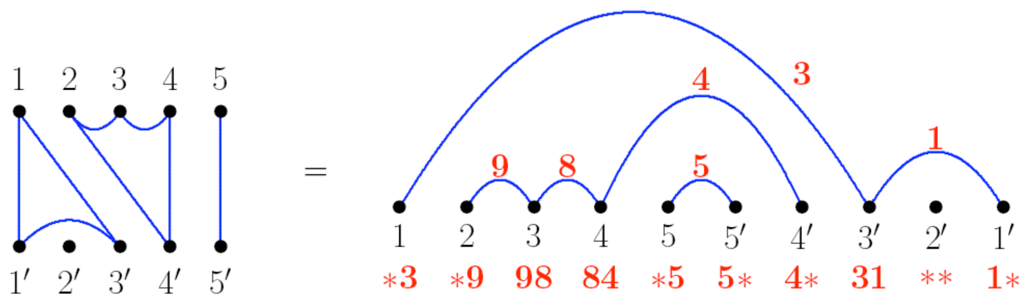




$$\sum_{r=0}^{k} \binom{k}{r}^2 r!$$

Planar Diagrams

V. Jones, 1994
L. Kauffman 1987

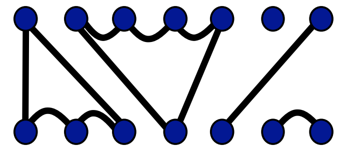


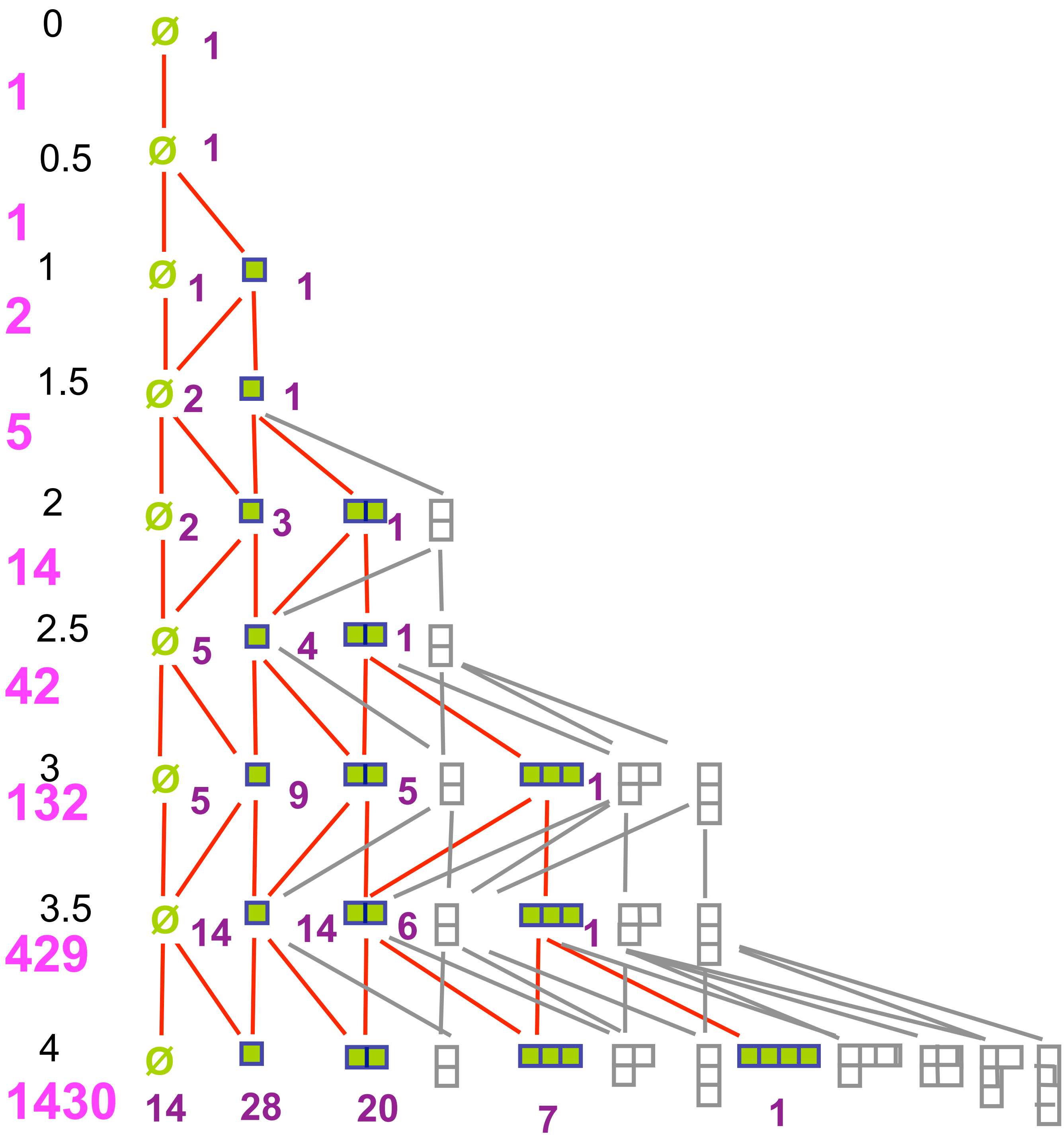
vertex	step	insertion		vertex	step	insertion	
1	$\frac{1}{2}$	$*$	$\emptyset$	$1'$	$\frac{1'}{2}$	$*$	$\emptyset$
	1	$+3$	[3]		$1'$	-1	[1]
2	$1\frac{1}{2}$	$*$	[3]	$2'$	$1\frac{1'}{2}$	$*$	[1]
	2	$+9$	[3|9]		$2'$	$*$	[1]
3	$2\frac{1}{2}$	-9	[3]	$3'$	$2\frac{1'}{2}$	$+1$	$\emptyset$
	3	$+8$	[3|8]		$3'$	-3	[3]
4	$3\frac{1}{2}$	-8	[3]	$4'$	$3\frac{1'}{2}$	$*$	[3]
	4	$+4$	[3|4]		$4'$	-4	[3|4]
5	$4\frac{1}{2}$	$*$	[3|4]	$5'$	$4\frac{1'}{2}$	$*$	[3|4]
	5	$+5$	[3|4|5]		$5'$	-5	[3|4|5]

No crossings ⇒ always get ≤ 1 row partitions

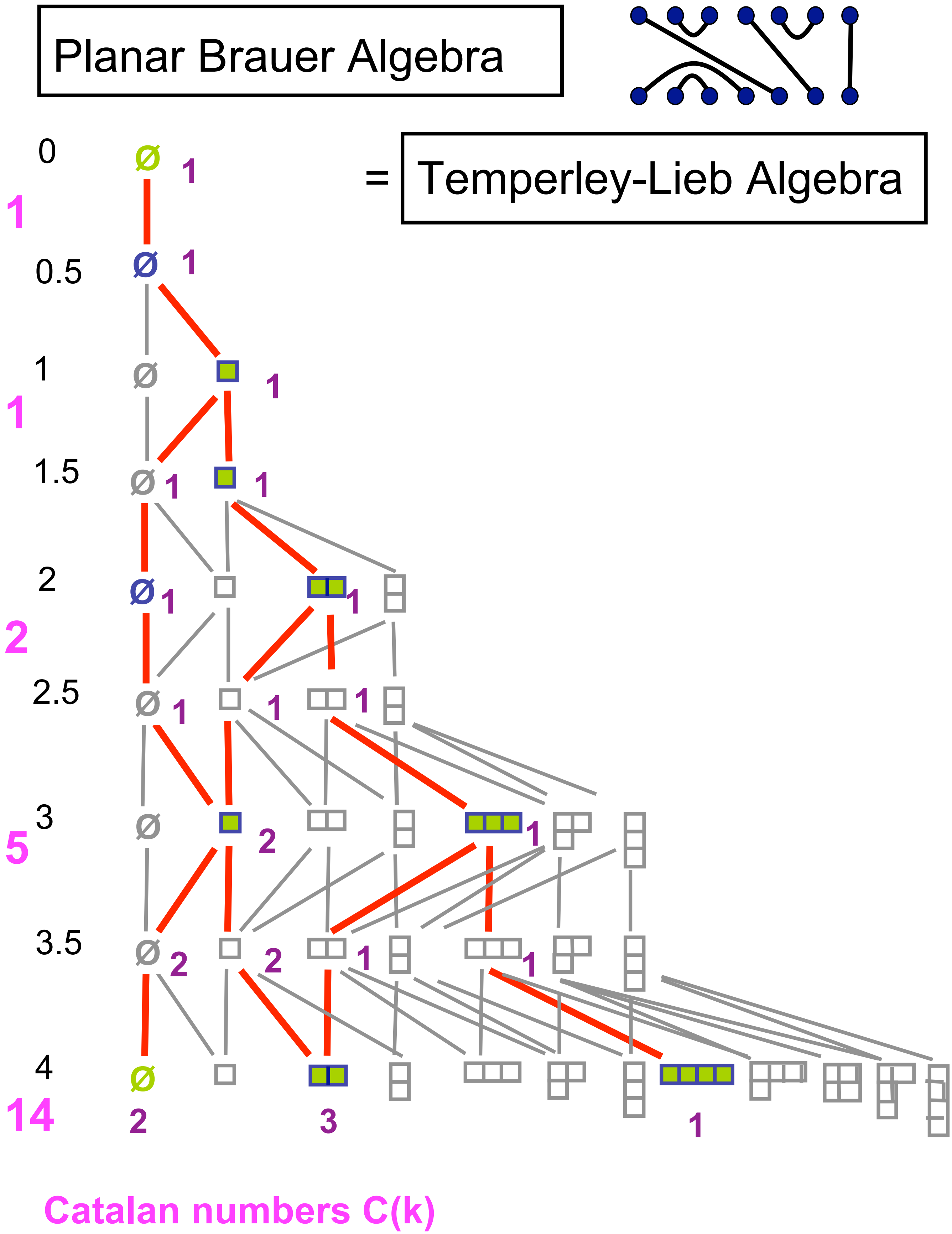
Special case of Theorem 3.2 in Chen-Deng-Du-Stanley-Yan

Planar Partition Algebra

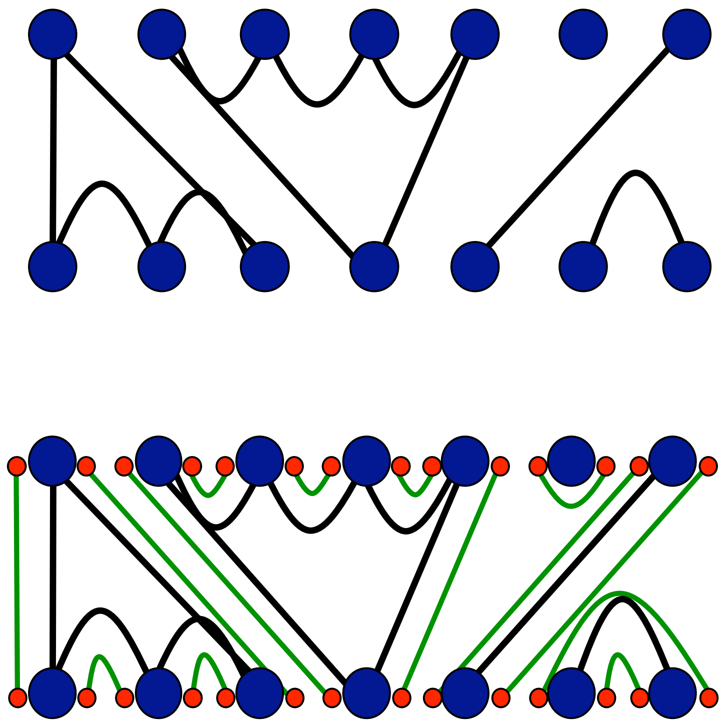




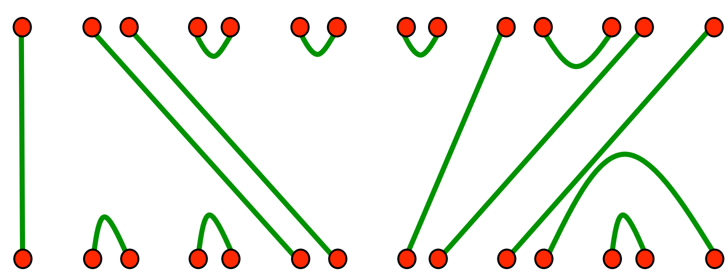
Catalan numbers C(2k)


Planar Brauer Algebra
= Temperley-Lieb Algebra
0
0.5
1
1.5
2
2.5
3
3.5
4
1
1
2
5
14
Catalan numbers C(k)

Bijection: Planar $P_k \leftrightarrow$ Temperley-Lieb T_{2k}

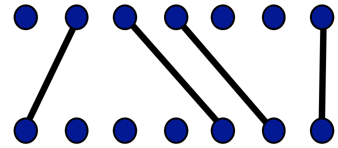


In Planar P_7

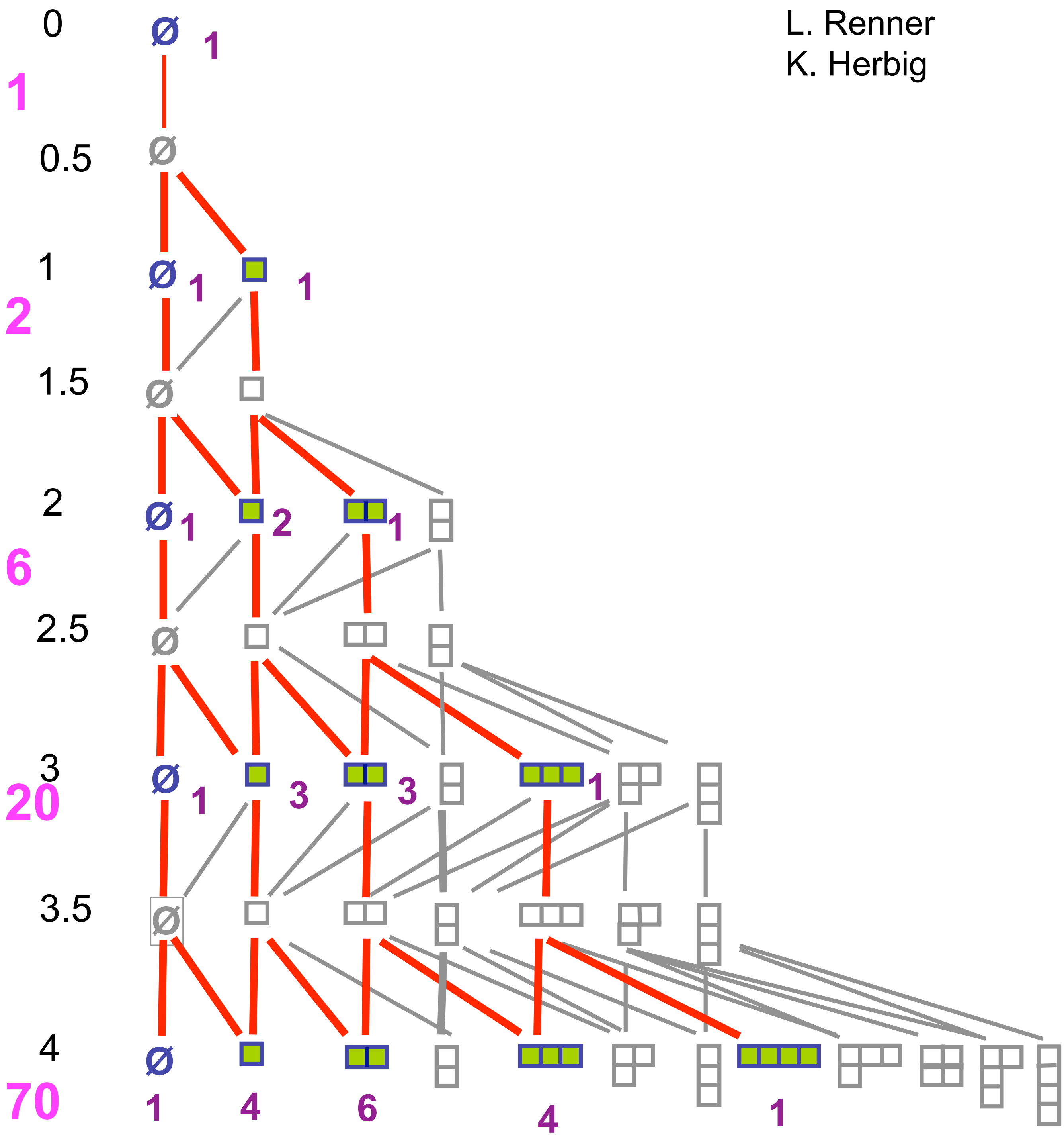


In T_{14}

Planar Rook Monoid Algebra



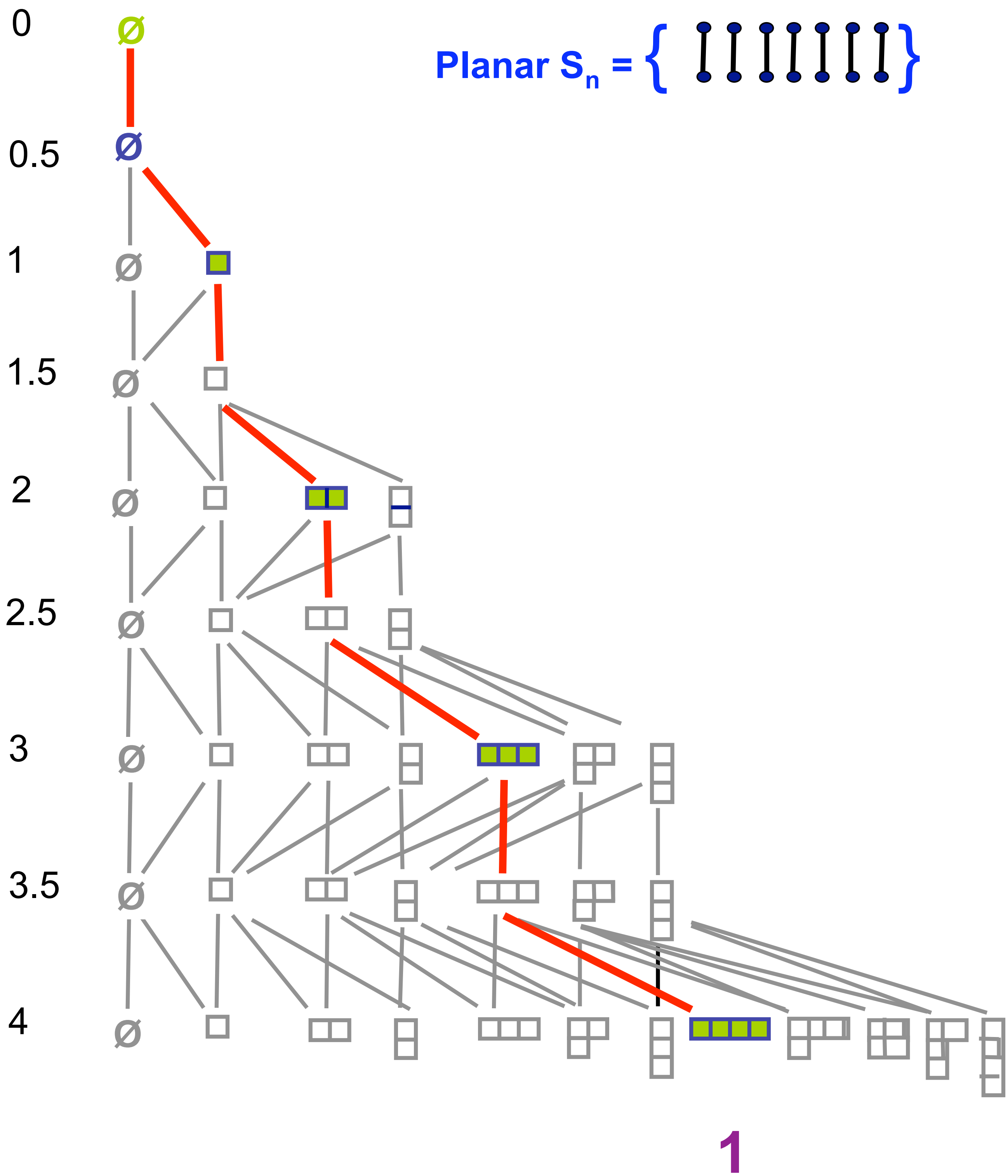
L. Renner
K. Herbig



$$\binom{2k}{k} = \sum_{\ell=0}^{k} \binom{k}{\ell}^2$$

Planar Symmetric Group?

Planar Symmetric Group

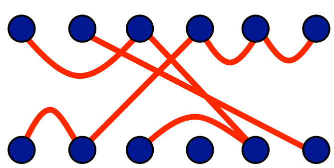


Propagating Number

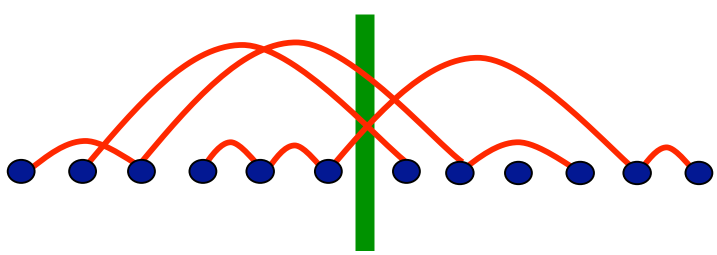
$$pn(d) = \#\left\{ \begin{array}{l} \text{blocks in d that contain a vertex} \\ \text{In top } \{1,\ldots,k\} \text{ and bottom } \{1',..,k'\} \end{array} \right\}$$

$$pn(ab) \leq pn(a)\, pn(b)$$

$$I_r = \text{span}\{\, d \mid pn(d) \leq r \,\}$$ ***an ideal***



pn = 3



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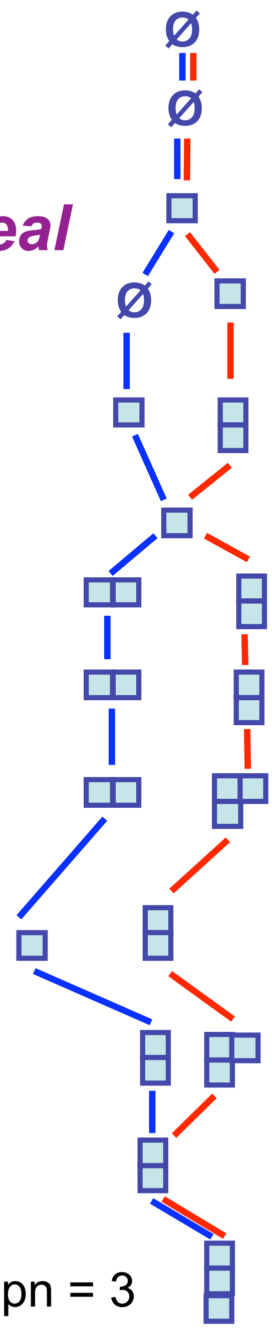
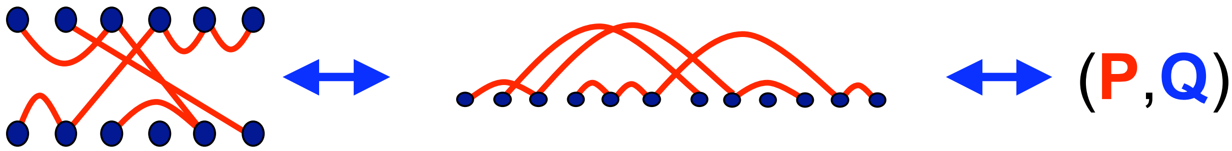


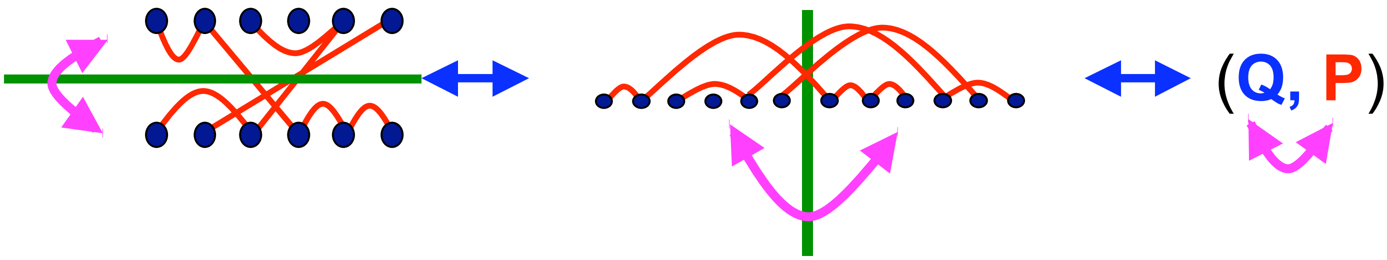
pn = 3

Symmetry Property

Growth diagrams
Fomin, Roby



Transpose the diagrams

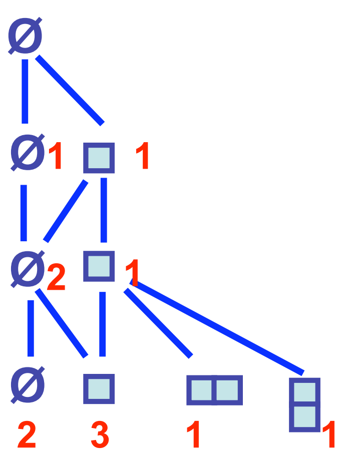


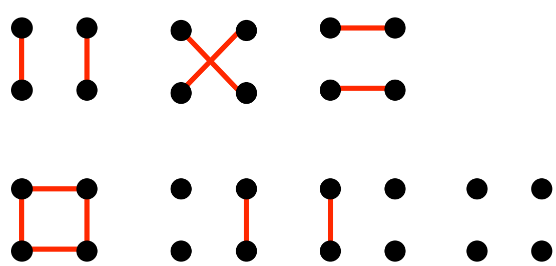
$\Rightarrow$

$$\#\left\{\text{symmetric diagrams}\right\} = \#\left\{(P,P)\right\} = \#\left\{\text{paths to level k}\right\}$$

$$= \sum_{\lambda} \dim(P_k^{\lambda})$$

eg. k=2

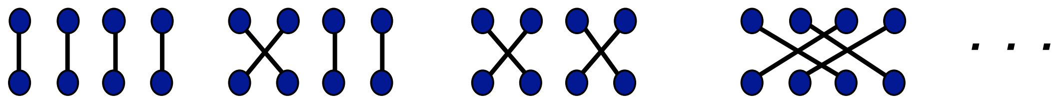




2 + 3 + 1 + 1 = 7

A Multiplicity-Free Model for S_k

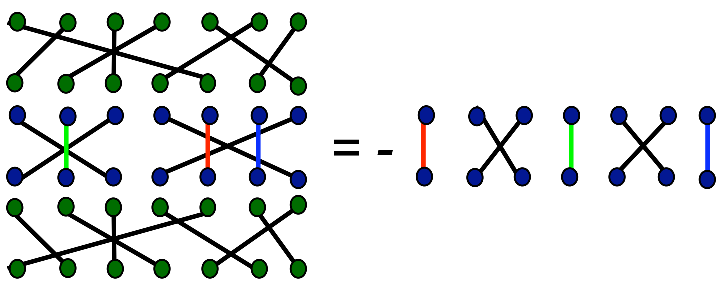
Symmetric permutations are involutions ($t^2 = 1$):



$M = \text{span-}\{$ *involutions in S_k* $\}$

$\sigma \cdot t = \mathbf{sign}(\sigma, t)\ \sigma t \sigma^{-1}$

$\mathbf{sign}(\sigma, t) =$ sign of the permutation of fixed points of σ



Theorem(J Saxl, 1981)

$$M \cong \bigoplus_{\lambda \vdash k} S_k^{\lambda}$$

(Each irreducible appears with multiplicity 1)

A Multiplicity-Free Model for Diagram Algebras

$N = \text{span-}\{$ *symmetric diagrams in D_k* $\}$

$$d \cdot s = \mathbf{sign}(d,s)\ dsd^T$$

$$\mathbf{sign}(d,s) = \begin{cases} 0, & \text{if } \mathrm{pn}(dsd^T) < \mathrm{pn}(s) \\ \text{sign of the permutation of fixed blocks of } d & \mathrm{pn}(dsd^T) = \mathrm{pn}(s) \end{cases}$$

$$\Longrightarrow \qquad N \cong \bigoplus_{\lambda} D_k^{\lambda}$$

Remarks

- No known uniform algebraic proof for D_k
- For S_n-$P_k(n)$ can use Frobenius reciprocity and the Schur functor
- Have checked the characters for all the diagram subalgebras for small k
- RSK Insertion proof on characters for S_k and possibly $P_k(n)$. (M Decker)

$$\sum_{\lambda} \chi^{\lambda}(d) = \sum_{\lambda} \sum_{P_\lambda} \mathrm{wt}(P_\lambda, d)$$

RSK → $$= \sum_{t} \mathrm{wt}(t,d) = \sum_{t} \mathrm{saxl}(t,d) = \chi_N(d)$$

Some references:

- T Halverson and T Lewandowski, **RSK Insertion For Set Partitions and Diagram Algebras**, *Electronic J Combinatorics*, **11(2)** R24, 2004-2005.

- W Chen, E Deng, R Du, R Stanley, C Yan, **Crossings and Nestings of Matchings and Partitions**, *Trans. Amer. Math. Soc*., to appear.

- T Halverson, A Ram, **Partition Algebras**, *European J. Combin*., **26** (2005), 869--921.

- Posters at this Meeting (Undergraduate Research Poster Session)
    - M Decker, RSK Insertion and a Model for the Symmetric Group
    - K Herbig, The Planar Rook Monoid