

Math 131/135/194, Fall 2004

## Applied Calculus

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## Review Problems for the Final Exam

## Problem 1

../DE/102b.tex

Consider the pair of differential equations

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -5x - y\end{aligned}$$

Starting at the initial condition  $y(0) = 0$  and  $x(0) = -5$ , compute  $x(t)$  and  $y(t)$  at times  $t = 0.01$  using a step size of 0.01

## Problem 2

../DE/106.tex

Consider the differential equation

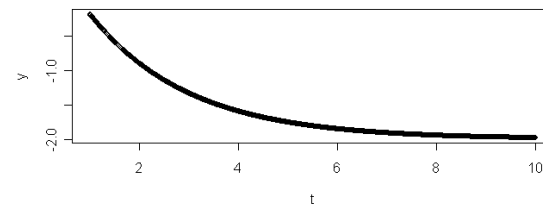
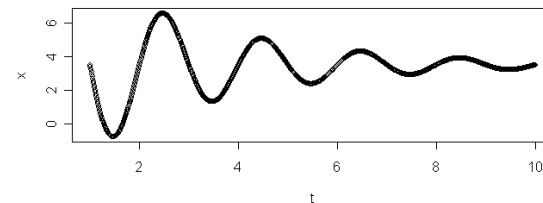
$$\frac{dy}{dt} = -ay + m + bxy \quad (1)$$

where  $x$  refers to the number of rabbits per acre and  $y$  refers to the number of foxes per acre (there is another equation for  $\frac{dx}{dt}$  but that doesn't concern us here).

1. Describe in words the biological or physical factors in this model that determine the rate of change of the fox population.
2. Given that time  $t$  is measured in years, give the units for each of the constants  $a$ ,  $m$ , and  $b$ .

## Problem 3

../DE/107.tex

The graphics below give the solutions  $x(t)$  versus  $t$  and  $y(t)$  versus  $t$ .Draw axes for the  $x, y$  phase plane and on those axes draw the trajectory of the system. Make sure to label your axes and place numbered tick marks. Mark both the initial condition and your guess about where the state goes as  $t \rightarrow \infty$ .

### Problem 4

../DE/103.tex

Match the eigenvalues to the solution: Write the letter next to the appropriate graph and say, in words, what features of the graph cause you to make the match.

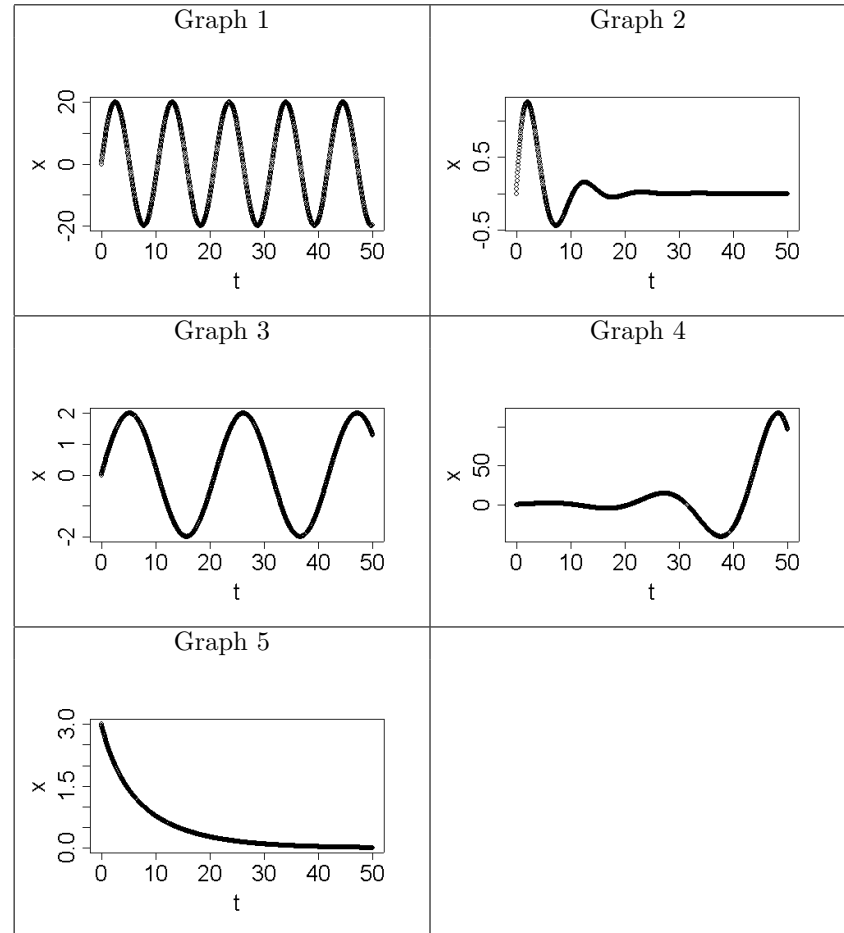
(a)  $0.1 \pm 0.3i$

(b)  $0 \pm 0.3i$

(c)  $-0.2 \pm 0.6i$

(d)  $-0.2 \pm 0.1$

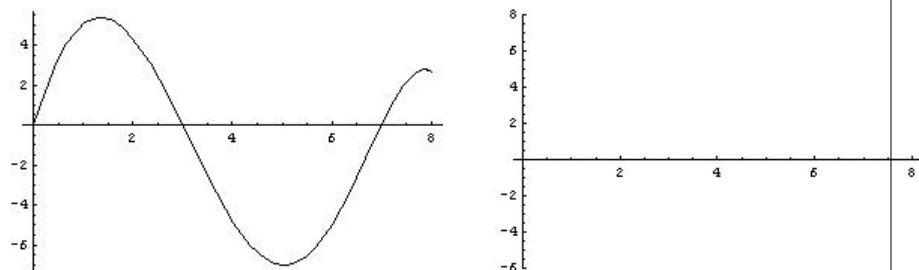
(e)  $0 \pm 0.6i$



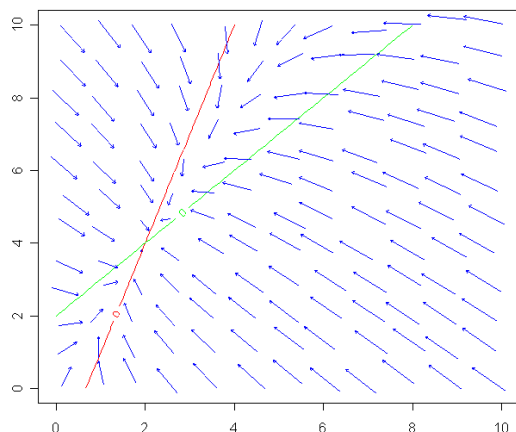
### Problem 5

../DE/109.tex

1. Pictured below is a plot of  $\frac{dp}{dt}$  as a function of  $p$ . Sketch  $p(t)$  on the axes to the right if  $p(0) = 6$



2. Pictured below is the phase diagram for a differential equation describing  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ .



- (a) Plot a plausible trajectory for  $(x(t), y(t))$  starting at the initial value  $(8, 6)$ .
- (b) Plot the solutions  $x(t)$  versus  $t$  and  $y(t)$  versus  $t$ , corresponding to your trajectory in part (i).

### Problem 6

../misc/2.tex

Consider the set of differential equations

$$\begin{aligned}\frac{dS}{dt} &= b - cSI + dR \\ \frac{dI}{dt} &= cSI - fI \\ \frac{dR}{dt} &= fI - dR\end{aligned}$$

where  $S$  refers to the number of people susceptible to a disease,  $I$  is the number infected, and  $R$  is the number recovered from the disease.

- (a) Describe in words the biological features of this model.
- (b) Given that time  $t$  is measured in days, give the units for each of the constants  $b$ ,  $c$ ,  $d$  and  $f$ .

### Problem 7

../misc/3.tex

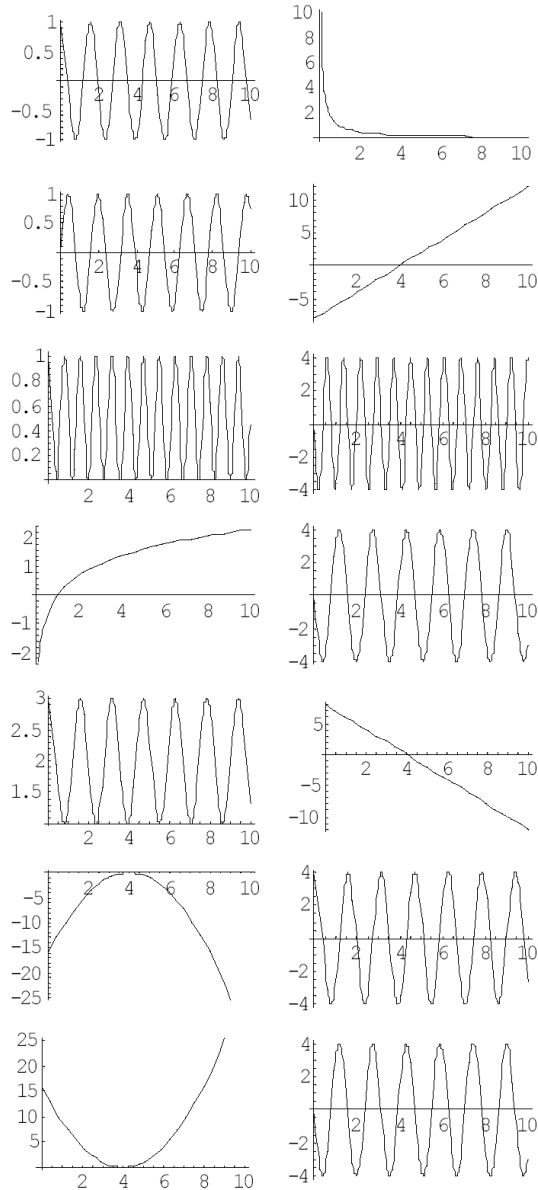
An article in the New York Times in Dec. 2003 reported that the Earth's magnetic field has weakened by 10% in the past 150 years. What is the half-life of the magnetic field if this is true, assuming an exponential model?

### Problem 8

../misc/4.tex

#### Graphs of functions and their derivatives

In the left column are the graphs of several functions; in the right are their derivatives. Match them up. Draw lines between the corresponding pairs of graphs.



### Problem 9

../misc/5.tex

#### Derivatives from Tables

The table below gives values of a function  $f(x)$ . Give estimates for the first and second derivatives of  $f$  at the point  $x_0 = 2$ .

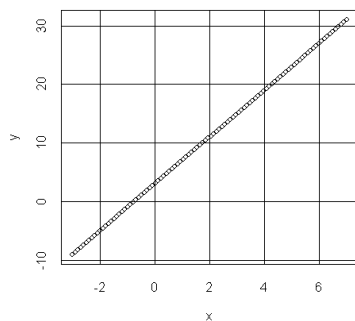
$x$	$f(x)$
0.0	4.000
0.2	3.608
0.4	3.264
0.6	3.016
0.8	2.912
1.0	3.000
1.2	3.328
1.4	3.944
1.6	4.896
1.8	6.232
2.0	8.000
2.2	10.248
2.4	13.024
2.6	16.376
2.8	20.352
3.0	25.000
3.2	30.368
3.4	36.504
3.6	43.456
3.8	51.272
4.0	60.000

### Problem 10

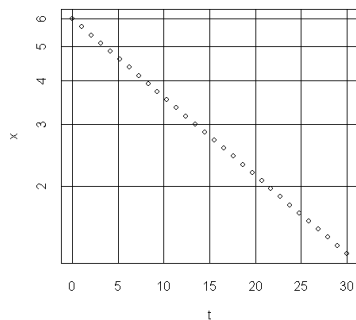
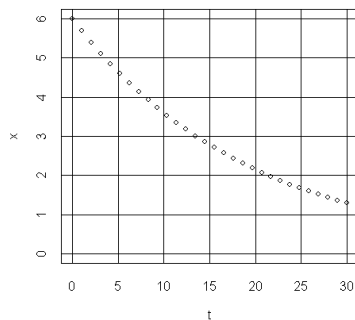
../misc/6.tex

In each part below, you are given a plot or plots of a function  $y(x)$  or  $x(t)$ . In each case, write down a formula for the  $f(x)$ . When the formula involves one or more parameters, write down the parameters symbolically, then give numerical values for the parameters. For example, if you were given the graph of a function that always has the value 8, you would write  $f(x) = c$ , where  $c = 8$ . Include a description in words of how you selected the function type and how you estimated the parameters, e.g., "The slope gives the parameter b."

(a)



(b) Both of the following graphs are depictions of the same function.



### Problem 11

../linear-algebra/1.tex

Calculate the cosine of the angles between the following pairs of vectors:

1.  $(1, 0)$  and  $(0, 1)$
2.  $(1, 1)$  and  $(0, 1)$
3.  $(1, 1, 1)$  and  $(1, 1, 0)$
4.  $(1, 1, 1)$  and  $(1, 0, 0)$

### Problem 12

../linear-algebra/3.tex

Calculate the lengths of the following vectors:

1.  $(1, 0)$
2.  $(1, 1)$
3.  $(3, 4, -2, 6)$

### Problem 13

../linear-algebra/2.tex

Find the projection of  $\vec{b}$  onto  $\vec{u}$  for the following vectors. Also, calculate the residual from the projection.

1.  $\vec{b} = (1, 1)$  onto  $\vec{u} = (1, 0)$   
in
2.  $\vec{b} = (1, 1, 1)$  onto  $\vec{u} = (1, 1, 0)$   
in

### Problem 14

../linear-algebra/4.tex

Write down some vectors that are perpendicular to the following:

1. One vector perpendicular to  $(1, 1)$
2. Two vectors perpendicular to  $(1, 1, 1)$
3. Three vectors perpendicular to  $(3, 4, -1, 2)$

### Problem 15

../linear-algebra/5.tex

We have solved the following system of equations in order to minimize the residual:

$$\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \approx \begin{pmatrix} 6 & -2 & 3 \\ 7 & 1 & 2 \\ 4 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (2)$$

1. For each of the following possible candidate solutions, calculate the residual.
  - $(1, -1, 3)$
  - $(2, 1, -3)$

- $(2, -2, 2)$
2. Which of the candidate solutions gives the shortest residual?
  3. Show why the candidate with the shortest residual is not, in fact, the least squares solution.

**Problem 16**

../linear-algebra/6.tex

Here is a system of linear equations

$$\begin{pmatrix} 30 \\ 45 \end{pmatrix} = \begin{pmatrix} 6 & 8 & -2 \\ 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3)$$

1. Which two of the following candidate solutions are correct:
  - (a)  $(5, 1, 4)$
  - (b)  $(6, 0, 9)$
  - (c)  $(6, 1, 3)$
2. Given the two solutions, generate a third, different solution and confirm that this is indeed an exact solution.
3. Using geometrical terms, explain why, in this case, there is more than one exact solution.

**Problem 17**

../linear-algebra/7.tex

Consider the vector  $(1, -3, 2, 4)$ . Give a complete specification of the subspace that is orthogonal to the vector.

**Problem 18**

../linear-algebra/8.tex

The following two vectors describe a subspace of two dimensions:  $(2, -1, -1)$  and  $(-3, 5, -2)$ . Say which of the following vectors is orthogonal to the subspace.

1.  $(3, 2, 1)$
2.  $(-1, 1, 1)$
3.  $(2, 4, 4)$
4.  $(1, 2, 0)$

**Problem 19**

../linear-algebra/102.tex

For  $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ , find

1. the  $x$  that brings  $\vec{v}_1 x$  as close as possible to  $\vec{b}$ .
2. the residual vector
3. the angle between  $\vec{v}_1$  and the residual vector

**Problem 20**

../linear-algebra/104.tex



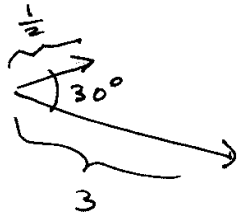
On the above graph (2 copies are provided to give you enough space), draw

1.  $-\vec{v}$
2.  $2\vec{v} + \frac{1}{2}\vec{w}$
3.  $\vec{v} - \vec{w}$

**Problem 21**

../linear-algebra/104b.tex

In the following graph, the lengths of two vectors are indicated, as is the angle between them.



Find  $\vec{v} \cdot \vec{w}$ .

**Problem 22**

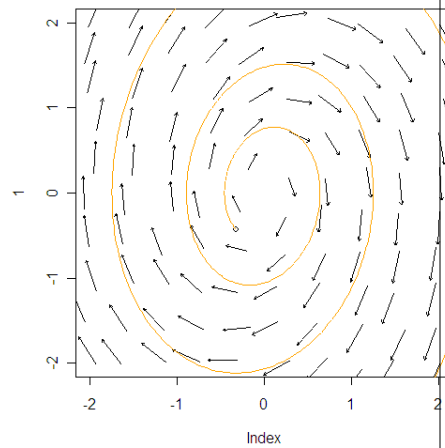
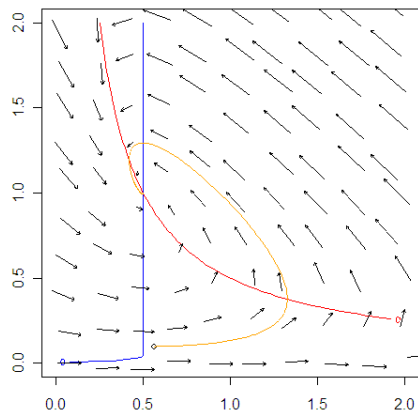
../linear-algebra/105.tex

For the function given below, find the gradient vector, and then evaluate it at the specified point.

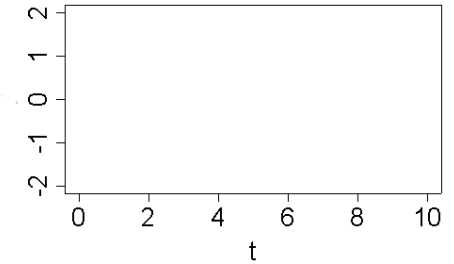
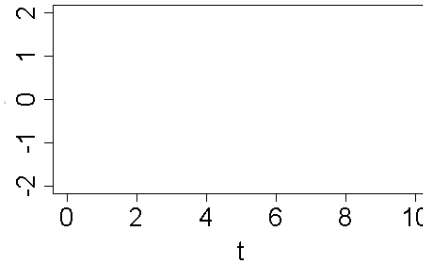
$$7 + 19x - 3y + x^2y + 9xy^7 \text{ at } (x = 1, y = -1)$$

**Problem 23**

../DE/105.tex



For each of the trajectories shown, sketch out the solution graph  $x(t)$  versus  $t$ .



**Problem 24**

../DE/111.tex

The following graph shows nullclines, flows, and trajectories. Unfortunately, the evil professor who made the graph introduced inconsistencies, for example a trajectory from one system but a flow field from another system. Next to the graph, explain what are the inconsistencies, circling features in the graph as appropriate.

