

FINDING AND CHARACTERIZING UNSTABLE FIXED POINTS BY CONTROLLING SYSTEM DYNAMICS

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We study time series data in order to understand better the underlying dynamics of a system. Unfortunately, many time series do not contain the information we seek in an accessible form, particularly in biological systems which habitually change during the course of an experiment. By using techniques originally developed for controlling chaotic systems, one can enhance the information contained in time series regarding the existence of hidden, unstable periodic orbits. This makes it possible, for example, to track drifts in a system's dynamics and to construct effective control schemes for stabilizing such orbits. Even when the system is under control, the characteristics of the unstable periodic orbits of the uncontrolled system can be estimated.

The important question ... may not be decidable on the basis of time series data.
-Thomas Schreiber at "Chaos in Brain?", 10 March 1999

All these questions are completely different if you are able to perturb the system.
-ibid.

1 Introduction

Some things are obvious and others are not. One of the tasks of time series analysis is to render clearly patterns that are not obvious in the raw data. There are examples of spectacular successes: by quantifying the energy in different bands of the EEG we can turn a complicated squiggle into a more simply interpreted set of numbers; through signal averaging we can extract a small evoked potential from a large amount of noise; by computing the correlation dimension, we can clearly demonstrate the deterministic chaotic nature of a signal that may look to the eye just like noise.

The reader may balk at that last example. Clearly there are cases where the correlation dimension of a chaotic signal is obviously different from the correlation dimension of noise, and where that noise is visually indistinguishable from the signal. But the history of chaotic analysis of the EEG is full of examples where the correlation dimension showed merely that the EEG is different from *white* noise; this is often completely obvious from visual examination of the raw time series, even to an untrained eye.

Faced with a more difficult test, say, distinguishing the EEG from spectrally matched noise, the correlation dimension has not proven to be such a success. In many cases there is found to be little difference between EEG and other neurological signals and spectrally matched "surrogate data."¹ In other cases the difference found is one that is entirely obvious to visual comparison of the signal and the surrogate

data.

In the end, much of the sophisticated nonlinear time series analysis we do does not have the effect of making things obvious that are not obvious in the raw data, but instead serves to quantify in abstract and not readily interpretable ways patterns that are obvious to the eye. Indeed, in many circumstances quantities such as the correlation dimension are unsuccessful at detecting differences that are obvious by eye. The several examples presented in the workshop where the correlation dimension indicates EEG changes not visible to the eye even of an experienced clinician illustrates another problem: we find it unsatisfying and suspect to deal with an abstract and and somewhat mysterious number.

One direction that time series research has moved in has been to develop yet more complicated and — we hope — more powerful techniques for extracting information from time series. We have seen some substantial moves in this direction, such as the advent of nonlinear-prediction methods for detecting nonlinearities and the development of new specific forms of surrogate data². At the “Chaos in Brain?” workshop, promising new work was presented on quantifying the synchrony between signals as a way of measuring coupling between neural assemblies in the brain.

In this communication, I would like to point in another direction: the possibility that by manipulating the systems we study we can make time series that are easy to analyze. The particular instance that I will discuss concerns the detection and characterization of unstable fixed points and periodic orbits.

1.1 Unstable Fixed Points in the History of Chaos

The theoretical link between chaos and unstable fixed points or unstable periodic orbits^a has a history as long as that of the concept of chaos itself. The original work by Poincaré on the three-body problem at the turn of the last century developed the idea of irregular but deterministic dynamics arising from the homoclinic tangle of stable and unstable manifolds emerging from saddle-type fixed points. (See Alligood, Sauer and Yorke³ for a nice introduction.) The re-introduction of chaos in the mid-1970s in terms of one-dimensional finite-difference equations was prompted largely by examination of successive period-doubling bifurcations (in which a stable periodic orbit loses stability in favor of an orbit whose period is twice as long) and also the development of theory that shows that the existence of a period-3 orbit implies the existence of orbits of all periods.

In nonlinear time series analysis there has been a long-standing theme of research involving the detection of unstable periodic orbits and ways of characterizing dynamical systems in terms of such orbits. This approach received a tremendous boost with the proposal of ways to use feedback control to stabilize such orbits: chaos control.

Soon after introduction of the theory of chaos control, the control was implemented experimentally. Possibly the first implementation was in a magneto-elastic ribbon⁴ where the dynamics were known to be well described as a low-dimensional

^aIf a system has a periodic orbit of period T , that system is closely related to a second system that has a fixed point. This second system is constructed by taking measurements from the first system once every T time steps.

finite-difference equation. Shortly thereafter, a modified form of chaos control was demonstrated in experiments involving heart tissue⁵ and in brain tissue⁶.

The application of chaos control in brain tissue was exciting for a number of reasons:

- Control immediately suggests the possibility of clinical applications; it shows that you can change the behavior of the system, perhaps in desirable ways.
- Control dramatically increases the directness of the link between data and conclusions. In previous work on chaos and the brain, one needed to apply somewhat sophisticated data analysis to get hints of the phenomena. In chaos control, the existence of a periodic orbit is immediately implicated by even casual examination of the data.
- The use of control suggests a new family of techniques for studying systems. Rather than merely observing the system dynamics, one perturbs the dynamics in ways that make the system much easier to study. (This idea predates chaos control; it can be seen in some early biological chaos work, notably that of Guevara et al.⁷.)
- The dynamical structure that underlies chaos control — the unstable periodic orbit — is conceptually quite simple and is easily envisioned to exist in a variety of systems, even those that are not chaotic. It is also a structure that might be seen even when substantial amounts of noise are involved in the system.

The experimental work involving the application of theoretical chaos control concepts to brain tissue is subject to differing interpretations. For example, Christini and Collins⁸ showed how a non-deterministic mechanism can account for the detailed results of the brain tissue control experiments. They demonstrated that many of the the patterns seen in the brain-tissue experiments can be replicated by simply truncating random numbers in a manner consistent with the controller dynamics.

The Christini and Collins work seems to be a plausible and sensible explanation for many of the observed experimental results. I do not wish to challenge it. But I would like to offer yet another explanation for the observed chaos-control results that suggests ways that experimenters can manipulate their systems in order to exploit one of the advantages of chaos control — the direct line between data and conclusions.

1.2 Unstable Fixed Points and Periodic Orbits

The relationship of fixed points to chaotic dynamics is not the only reason for interest in them. It is unclear what role, if any, chaos plays in complex neurological systems.^b But fixed points represent a simple form of dynamics that is potentially accessible even from rather coarse measurement.

^bThis is not to say that nonlinear time series analysis techniques motivated by chaos are not important. There is good evidence in other chapters of this volume that quantities such as the correlation dimension can provide some clinical insight into the EEG.

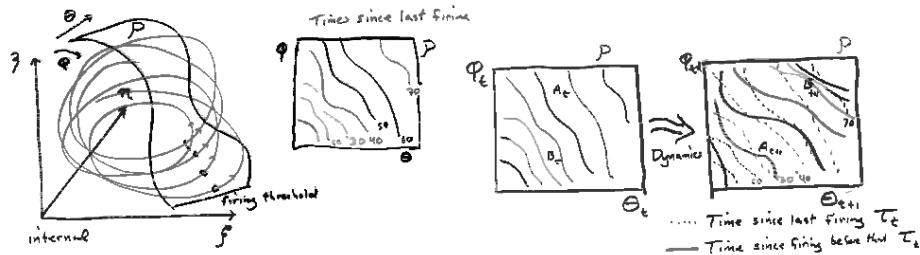


Figure 1. Left: The state of the system evolves in some state space whose components, ξ , η , and ζ , are unmeasurable to us. A measurable spike occurs whenever the state crosses some surface labelled \mathcal{P} ; the interspike interval records the time between successive piercings of this surface \mathcal{P} . Right: Pairs of successive interspike intervals allow a representation of the discrete-time dynamics from \mathcal{P} to itself, even when the position on \mathcal{P} is never measured directly.

Consider for example the kind of measurement often made in neurophysiology: the time of an action potential. Figure 1 shows a highly schematic drawing of the evolution of the state of a neurological system. The variables ξ , η , and ζ are not directly accessible to measurement and the state evolves along a trajectory that we cannot measure. However, we can measure the time of occurrence of an action potential. In the schematic state space of Fig. 1 the action potential occurs when the trajectory pierces a locus of points in the state space, shown as the sheet labeled \mathcal{P} . This sheet is a section through the space, a Poincaré section. For example, in an integrate-and-fire model, the time of firing is set by the amount of charge that accumulates on the cell membrane. Taking one of the abstract variables, say ζ , to represent this charge, the cell will fire when ζ reaches the threshold level. This locus of points $\zeta = \text{threshold}$ corresponds to a plane in the (ξ, η, ζ) -space.

Since we cannot measure ξ , η , and ζ directly, this schematic view may seem an irrelevancy. But each point on the \mathcal{P} -surface in Fig. 1 is connected by a deterministic trajectory to a unique previous point on the surface; the time taken to follow the trajectory from one piercing of the \mathcal{P} -surface to the previous piercing is the time between action potentials — the interspike interval — something we can measure.

For each point on the \mathcal{P} -surface there is a given interspike interval. This means that when we measure the interspike interval we are measuring where on the \mathcal{P} -surface the state is. Generically, the possible positions on \mathcal{P} corresponding to a given interspike interval is a curve, or perhaps multiple curves. We'll call these curves for any given value of the interspike interval to be the isochrone for that interspike interval. So the measurement of a single interspike interval gives some, but not complete information about the position of the state on \mathcal{P} at the instant when the interspike interval measurement was made. Some schematized isochrones are shown in Fig. 1.

Since the hypothetical dynamics are hypothesized to be deterministic, each point on \mathcal{P} corresponds to a unique point or set of points that are the possible values of the state ξ , η , and ζ on \mathcal{P} during the previous piercing of \mathcal{P} . A consequence of this is that each isochrone on \mathcal{P} is mapped by the dynamics to another curve on \mathcal{P} , the

image of the isochrone. By using both the current ISI and the previous ISI, we can localize the state to the intersection of the two corresponding isochrones curves on \mathcal{P} : the isochrone and the image of the isochrone. Thus, the 2-dimensional location of the successive piercings of \mathcal{P} can be represented by the sequence

$$\cdots \rightarrow \tau_{t-1}, \tau_t \rightarrow \tau_t, \tau_{t+1} \rightarrow \tau_{t+1}, \tau_{t+2} \rightarrow \cdots$$

We will call the (τ_t, τ_{t+1}) -plane the \mathcal{T} plane.

One problem with this view is that the continuous-time state space need not be limited to the 3 variables ξ, η, ζ . Indeed, there may be a very large number of variables involved, perhaps 10, perhaps 10^{10} . When there are N variables in the continuous-time state space the Poincaré section \mathcal{P} of spike generation will be $N - 1$ -dimensional. In principle, the location of each piercing of \mathcal{P} can be represented with $N - 1$ successive interspike intervals. This is the idea of lag embedding.⁹

In the experimental work^{5,6} the 2-dimensional τ_t, τ_{t+1} plane was used to find and control fixed points. One interpretation of the success reported in this work is that the presumably high dimensional state of their tissue samples expressed itself in continuous-time dynamics that could be represented in 3 variables.

Another interpretation is that the projection of the dynamics from a high-dimensional space to the 2-dimensional representation \mathcal{T} did indeed involve a loss of information but that this lost information did not prohibit successful control. In particular, the lost information might make it appear that at different times there are qualitatively different and inconsistent dynamics at a single point in \mathcal{T} . An implication of either interpretation is that it will not be possible to find a single valued function, $f : \mathcal{T} \rightarrow \mathcal{R}$ that summarizes the dynamics.

In order to find f , if it does exist, one might for instance take all triplets $\tau_t, \tau_{t+1} \rightarrow \tau_{t+2}$ and use regression to find a functional relationship. In the experimental work^{5,6} another approach was taken. They examined short sequences of points $\tau_k, \tau_{k+1}, \tau_{k+2}, \tau_{k+3}, \tau_{k+4}, \tau_{k+5}, \dots$ which appeared to follow the pattern expected if the system's state at time k were near a fixed point in \mathcal{P} . The expected pattern was a pair of points (τ_k, τ_{k+1}) and (τ_{k+1}, τ_{k+2}) on a line of shallow slope between 1 and -1 , with the second point having $\tau_{k+1} \approx \tau_{k+2}$ placing it near the hypothetical fixed point (τ_*, τ_*) . This pair of points is to be followed by a series of points $(\tau_{k+2}, \tau_{k+3}), (\tau_{k+3}, \tau_{k+4}), \dots$ which fall on another, steeper line and are successively farther from (τ_*, τ_*) . The overall pattern is one of advance toward the fixed point and retreat from it.

In general, though, the motion near a fixed point does not consist of such an advance and retreat along two distinct lines. Instead, the sequence of points lies on a hyperbola (or a pair of hyperbolas for flip dynamics) not a pair of lines. A more general description is the function

$$\tau_{t+1} = a\tau_t + b\tau_{t-1} + c \tag{1}$$

that gives a linear approximation to the dynamics near the fixed point.

If the state is near a fixed point in \mathcal{P} at time k , then parameters a, b , and c of the dynamics in \mathcal{T} can be estimated from the 5-point sequence $\tau_k, \tau_{k+1}, \tau_{k+2}, \tau_{k+3}, \tau_{k+4}$ by solving for a, b , and c the simultaneous equations

$$\tau_{k+2} = a\tau_{k+1} + b\tau_k$$

$$\tau_{k+3} = a\tau_{k+2} + b\tau_{k+1}$$

$$\tau_{k+4} = a\tau_{k+3} + b\tau_{k+2}$$

The location of the fixed point can then be estimated as

$$\tau_{\star} = c/(1 - a - b).$$

Of course, such an estimation is of limited use unless one knows the time k at which the state was near the fixed point. Since we do not know k we need to find suitable k from the data themselves. We can do this using the following procedure:

1. Find a , b , c , and τ_{\star} for every 5-point sequence in the time series. For each $i = 1, 2, 3, \dots$, call $\tau_{i\star}$ the value of τ_{\star} estimated from the sequence starting at $k = i$.
2. Histogram the resulting sequence of $\tau_{i\star}$.
3. Examine the histogram for a sharp peak. The value of τ_{\star} for this peak is then the estimated location of the fixed point. The indices k_1, k_2, \dots that contributed the points in the histogram's peak are the times we estimate to be when the system's state was near the hypothetical fixed point in \mathcal{P} .
4. Extract the triplets that are close to the estimated times when the state is near the hypothetical fixed point and use them to fit the parameters a , b , and c using a least squares or other technique. That is, solve the overdetermined system

$$\tau_{k_1+2} = a\tau_{k_1+1} + b\tau_{k_1}$$

$$\tau_{k_1+3} = a\tau_{k_1+2} + b\tau_{k_1+1}$$

$$\tau_{k_1+4} = a\tau_{k_1+3} + b\tau_{k_1+2}$$

$$\tau_{k_2+2} = a\tau_{k_2+1} + b\tau_{k_2}$$

$$\tau_{k_2+3} = a\tau_{k_2+2} + b\tau_{k_2+1}$$

$$\tau_{k_2+4} = a\tau_{k_2+3} + b\tau_{k_2+2}$$

and so on

A modified form of this method, but described in quite different terms, was presented by So et al.¹⁰.

One serious problem with this approach is that it relies on multiple visits to near the fixed point in order to produce the peak in the histogram. The method would not work, for instance, for a periodic signal despite the fact that dynamics that lead to periodic trajectories often contain fixed points.

2 PPF Control and Fixed Points

The control method described in ^{5,6}, called PPF control,^c consisted of applying timed stimuli to trigger firing. The control algorithm is as follows:

- Step 1: Define a line $A\tau_t + C$ called the “control line.”
- Step 2: After the next firing, take the ISI terminated by the firing to be τ_t . Compute

$$\tau_{max} = A\tau_t + C.$$

- Step 3: Wait until the next firing or until τ_{max} has elapsed since the firing in Step 2, whichever comes first. If τ_{max} is first, stimulate the tissue to trigger an immediate firing.
- Step 4: Go to Step 2 to continue the control.

The effect of this algorithm is to allow only those pairs (τ_t, τ_{t+1}) that lie on or below the control line $\tau_{t+1} = A\tau_t + C$. If a point would be above the control line, τ_{t+1} is truncated to put the point on the control line.

In the experimental work^{6,5} it was proposed to place the control line on the stable manifold of the fixed point τ_* . The idea is to have the control stimulus carry the state to the stable manifold, where the system’s natural dynamics will carry it to the fixed point.

Obviously, in order to carry out this strategy it is necessary to know the location of the fixed point and its stable manifold. This raises some questions:

- Since the stable manifold is estimated from the data, its position is somewhat uncertain. Can the control be successful if the control line does not lie on the stable manifold?
- What if the fixed point has no stable manifold? Is control possible?
- Suppose the time series is periodic. The underlying system may well have a fixed point, but the information provided by the time series will give no indication of its location. Yet in principle control should work if there is a fixed point. Can the control method work even when the location and characteristics of the fixed point are completely unknown?

3 The Dynamics of PPF Control

With control implemented, the dynamics of the system near the fixed point are

$$\tau_{t+1} = \min \begin{cases} a\tau_t + b\tau_{t-1} + c & \text{Natural dynamics} \\ A\tau_t + C & \text{Control line: stimulus used} \end{cases} \quad (2)$$

^cPPF control is the acronym for “Proportional Perturbation Feedback.” I believe this term is misleading — since the perturbation, as we will see, doesn’t involve a proportion — and propose in its place Procrustean Perturbation Feedback after mythical Greek innkeeper who, confronted with a guest who was too tall to fit in the available bed, truncated his guest.

These controlled dynamics are nonlinear, but in an important case they can be easily studied. Consider the *controller fixed point*, the value x_* such that $x_* = Ax_* + C$. This is the fixed point of the system if the control stimulus is used every iteration.

If the control stimulus is activated each iteration then the dynamics of the system are simply

$$\tau_{t+1} = A\tau_t + C.$$

The stability of x_* is then determined entirely by A which is *set by the experimenter*. The experimenter can guarantee stability of the controller fixed point by setting $|A| < 1$. For example, by setting $A = 0$ stability is assured and PPF control becomes what is called *demand pacing* in the medical terminology.

In order to assure that the stimulus will be active every iteration — thus creating the dynamics described in the previous paragraph — once the state is near (x_*, x_*) it suffices that the image of (x_*, x_*) under the natural dynamics be above the control line and that the same is true for at least a small neighborhood of (x_*, x_*) on the control line. So, we require

$$x_* < ax_* + bx_* + c.$$

This condition will be satisfied for a variety of different types of unstable fixed point types: saddles, nonflip saddles, flip and double-flip repellers, and spirals — so long as the proper relationship is established between x_* and τ_* . For example, for a non-flip saddle one requires $x_* > \tau_*$, while for a spiral the opposite condition holds. This means, remarkably, that stable control can be established even when *there is no stable manifold*. (The exception is a non-flip repeller, for which PPF control cannot create stability.)

When the necessary relationship between τ_* and x_* does not hold, the fixed point of the controlled dynamics will not be at the controller fixed point x_* and the dynamics of the controlled system may be stable periodic, unstable, chaotic, etc. For example, for a flip saddle a stable controller fixed point requires $x_* < \tau_*$. When x_* is slightly larger than τ_* , the controlled dynamics will have a stable period-2 orbit.

4 Identifying the Natural Dynamics while under Control

The above shows that the existence of a stable manifold is essentially irrelevant to the success of PPF control. This being the case, we do not need to estimate the location of the natural fixed point prior to the establishment of control; we need merely scan the control parameter over a range of plausible value. When x_* is sufficiently close to a fixed point and on the correct side of τ_* the controller fixed point will be stable. Since a bifurcation in the controlled system dynamics occurs when $x_* = \tau_*$, the precise location of τ_* can be found.

The imposition of control makes time series analysis a trivial task. When the controller fixed point is stable, the time series will (asymptotically) approach a single repeated value. When the period-doubling bifurcation occurs, the time series will alternate between two nearby values.

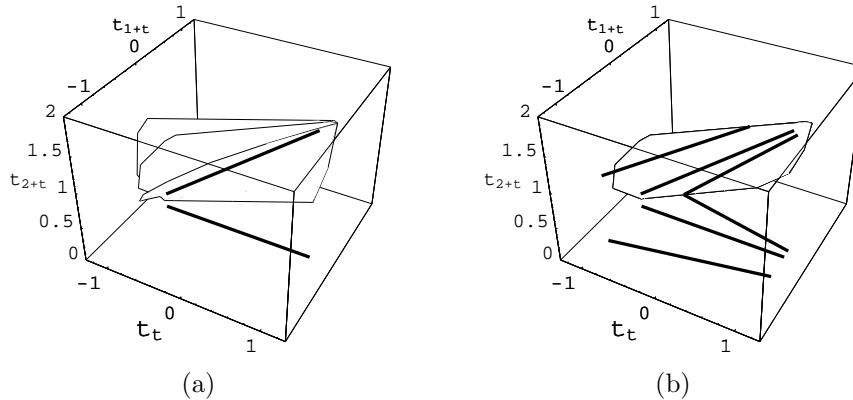


Figure 2. The linear dynamics near a fixed point can be represented as a plane in $\tau_{t+2}, \tau_{t+1}, \tau_t$ space. When the controlled system is in a period-two mode, all of the $\tau_{t+2}, \tau_{t+1}, \tau_t$ triples where τ_{t+2} was naturally terminated lie on a single line as shown in (a). There are many planes that are consistent with this line. By jiggling the control parameters, a unique plane is specified, as shown in (b).

The PPF-type of control, although easy to establish, is not very satisfactory in the sense that a control stimulus is being provided every iteration. The control becomes essentially periodic pacing of the tissue.

There are, of course, cases where entrainment of the system by periodic pacing is an important goal. For the present, though, our purpose is to create useful time series for analyzing the natural system. We have accomplished this objective to some extent — the time series resulting from the scanning of C is easily processed to find the bifurcation that will occur when x_* passes through τ_* . So, we have a method to find fixed points even when the natural system provides no hint of them, for example when the natural system has a periodic or almost periodic orbit.

Further insight into the natural system's dynamics near the fixed point can be had by studying time series from the system even while under control. This requires some small changes in the control algorithm, because obviously the system that obtains when the control stimulus is given every iteration, $\tau_{t+1} = A\tau_t + C$ tells nothing about the natural dynamics (except that a fixed point is nearby).

5 Characterizing Unstable Fixed Points While Controlling Them

For some types of unstable fixed points, for example flip saddles, a period-doubling bifurcation occurs when the controller fixed point x_* crosses the natural fixed point τ_* . Slightly above the bifurcation point, the dynamics alternate:

$$\begin{aligned} \tau_{t+1} &= a\tau_t + b\tau_{t-1} + c & \text{Odd } t: \text{ Natural Dyn.} \\ \tau_{t+1} &= A\tau_t + C & \text{Even } t: \text{ Control Dyn.} \end{aligned} \quad (3)$$

If we consider only those times where τ_{t+1} is the result of the natural dynamics, that is, every second ISI where the ISI was terminated naturally, the dynamics can

be rewritten:

$$\begin{aligned}\tau_{t+2} &= a\tau_{t+1} + b\tau_t + c = a(A\tau_t + C) + b\tau_t + c \\ &= (aA + b)\tau_t + (aC + c)\end{aligned}\tag{4}$$

This means that while under the period-doubled behavior, each natural ISI is a function only of the previous natural ISI. Note that there are only two parameters that enter into the dynamics of Eq. 4, $(aA + b)$ and $(aC + c)$. There are, however, three parameters that are needed to describe the natural dynamics, a , b , and c , and even though the experimenter sets A and C , we cannot find the three natural parameters if A and C are fixed. We are left with the situation depicted in Fig. 2a: the surface $\tau_{t+1} = a\tau_{t+1} + b\tau_t + c$ that describes the dynamics in $\tau_{t+2}, \tau_{t+1}, \tau_t$ space cannot be uniquely estimated because all of the points in the τ_{t+1}, τ_t plane lie on a line — the control line.

There is a simple solution to this problem: the experimenter can *jiggle* the control parameters A and C , varying them, for example, randomly over a small range. This spreads out the points in the τ_{t+1}, τ_t plane along several lines and uniquely defines the surface in the $\tau_{t+2}, \tau_{t+1}, \tau_t$ space as seen in Fig. 2b.

Another approach, detailed in Christini and Kaplan,¹¹ is useful when the unstable fixed point is of a type that does not involve a period-doubling bifurcation as x_* is varied. Setting x_* close to the fixed point, one can turn off the control for two or more intervals, obtaining a short sequence of ISIs that are terminated naturally. Linear regression can then be used on this sequence to find the parameters a , b , and c of the natural dynamics.

In summary, applying control stimuli to the system can simplify the dynamics in ways that make it trivial to understand the controlled system and which still allow important features of the uncontrolled system to show through. Without control, these natural features might well be invisible or inaccessible through time series analysis.

Acknowledgments

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