eg Confirm that $\psi_1(x) = |1\rangle = N_1 (2y) e^{-y^2/2}$

is an eigenfunction of the harmonic oscillator Hamiltonian.

Answer: Start by evaluating separate terms of the Hamiltonian, then combine them.

$$\frac{d\psi}{dx} = \frac{d\psi}{dy}\frac{dy}{dx} \quad \text{(chain rule)} \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{\alpha} \quad \left(\text{since } y = \frac{x}{\alpha}\right)$$

so $$\frac{d\psi}{dx} = \frac{1}{\alpha}\frac{d\psi}{dy}$$

and $$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{d}{dx}\left(\frac{1}{\alpha}\frac{d\psi}{dy}\right) = \frac{1}{\alpha}\frac{d\left(\frac{d\psi}{dy}\right)}{dx}$$

$$= \frac{1}{\alpha}\frac{d\left(\frac{d\psi}{dy}\right)}{dy}\frac{dy}{dx} = \boxed{\frac{1}{\alpha^2}\frac{d^2\psi}{dy^2}}$$

$$\frac{d\psi}{dy} = 2N_1\left[y(-y)e^{-y^2/2} + e^{-y^2/2}\right] = 2N_1\left[-y^2 e^{-y^2/2} + e^{-y^2/2}\right]$$

$$\frac{d^2\psi}{dy^2} = 2N_1\left[(-y^2)(-y)e^{-y^2/2} + e^{-y^2/2}(-2y) + (-y)e^{-y^2/2}\right]$$

$$= 2N_1 e^{-y^2/2}\left[y^3 - 2y - y\right] = 2N_1 e^{-y^2/2}(y^3 - 3y)$$

$$= 2N_1 y\, e^{-y^2/2}(y^2 - 3) = \boxed{\psi_1 (y^2 - 3)}$$

So $$\boxed{\frac{d^2\psi}{dx^2} = \frac{1}{\alpha^2}(y^2-3)\psi_1}$$

so $\hat{H}\psi_1 = -\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + \frac{1}{2}kx^2\psi_1$

$= -\frac{\hbar^2}{2m}\frac{(y^2-3)}{\alpha^2}\psi_1 + \frac{1}{2}k\alpha^2 y^2\psi_1$ (since $y = \frac{x}{\alpha}$)

$= \left[-\frac{\hbar^2}{2m\alpha^2}y^2 + \frac{3\hbar^2}{2m\alpha^2} + \frac{k\alpha^2}{2}y^2\right]\psi_1 = E_1\psi_1$

$\Rightarrow \boxed{E_1 = \left(\frac{k\alpha^2}{2} - \frac{\hbar^2}{2m\alpha^2}\right)y^2 + \frac{3\hbar^2}{2m\alpha^2}}$

But since an eigenvalue must be a real number, the coefficients of all non-constant terms must equal zero (if ψ_1 truly is an eigenfunction).

so $\frac{k\alpha^2}{2} - \frac{\hbar^2}{2m\alpha^2} \stackrel{?}{=} 0$

$km\alpha^4 - \hbar^2 \stackrel{?}{=} 0$ and $\alpha = \left(\frac{\hbar^2}{mk}\right)^{1/4}$

$\cancel{km}\left(\frac{\hbar^2}{\cancel{mk}}\right) - \hbar^2 = \hbar^2 - \hbar^2 = 0$ [YES!]

so $E_1 = \frac{3\hbar^2}{2m\alpha^2} = \frac{3\hbar^2}{2m}\sqrt{\frac{mk}{\hbar^2}} = \frac{3\hbar}{2}\sqrt{\frac{k}{m}} = \frac{3}{2}\hbar\omega$

agreeing with general formula: $E_v = (v+\frac{1}{2})\hbar\omega$

$= \frac{3}{2}\hbar\omega$ when $v = 1$

(!)