eg. Confirm that $\psi_1(x) = N_1 (2y) e^{-y^2/2}$

is an eigenfunction of the harmonic oscillator Hamiltonian.

Answer: Start by evaluating separate terms of the Hamiltonian, then combine them.

$$\frac{d\psi}{dx} = \frac{d\psi}{dy}\frac{dy}{dx} \quad \text{(chain rule)} \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{\alpha} \quad \left(\text{since } y = \frac{x}{\alpha}\right)$$

so $$\frac{d\psi}{dx} = \frac{1}{\alpha}\frac{d\psi}{dy}$$

and $$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{d}{dx}\left(\frac{1}{\alpha}\frac{d\psi}{dy}\right) = \frac{1}{\alpha}\frac{d\left(\frac{d\psi}{dy}\right)}{dx}$$

$$= \frac{1}{\alpha}\frac{d\left(\frac{d\psi}{dy}\right)}{dy}\frac{dy}{dx} = \boxed{\frac{1}{\alpha^2}\frac{d^2\psi}{dy^2}}$$

$$\frac{d\psi}{dy} = 2N_1\left[y(-y)e^{-y^2/2} + e^{-y^2/2}\right] = 2N_1\left[-y^2 e^{-y^2/2} + e^{-y^2/2}\right]$$

$$\frac{d^2\psi}{dy^2} = 2N_1\left[(-y^2)(-y)e^{-y^2/2} + e^{-y^2/2}(-2y) + (-y)e^{-y^2/2}\right]$$

$$= 2N_1 e^{-y^2/2}\left[y^3 - 2y - y\right] = 2N_1 e^{-y^2/2}(y^3 - 3y)$$

$$= 2N_1 y e^{-y^2/2}(y^2-3) = \boxed{\psi_1 (y^2-3)}$$

So $$\boxed{\frac{d^2\psi}{dx^2} = \frac{1}{\alpha^2}(y^2-3)\psi_1}$$

$$\text{so } \hat{H}\psi_1 = -\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + \frac{1}{2}kx^2\psi_1$$

$$= -\frac{\hbar^2}{2m}\frac{(y^2-3)}{\alpha^2}\psi_1 + \frac{1}{2}k\alpha^2 y^2\psi_1 \quad \left(\text{since } y = \frac{x}{\alpha}\right)$$

$$= \left[-\frac{\hbar^2}{2m\alpha^2}y^2 + \frac{3\hbar^2}{2m\alpha^2} + \frac{k\alpha^2}{2}y^2\right]\psi_1 = E_1\psi_1$$

$$\Rightarrow \boxed{E_1 = \left(\frac{k\alpha^2}{2} - \frac{\hbar^2}{2m\alpha^2}\right)y^2 + \frac{3\hbar^2}{2m\alpha^2}}$$

But since an eigenvalue must be a real number, the coefficients of all non-constant terms must equal zero (if ψ_1 truly is an eigenfunction).

$$\text{so } \frac{k\alpha^2}{2} - \frac{\hbar^2}{2m\alpha^2} \stackrel{?}{=} 0$$

$$km\alpha^4 - \hbar^2 \stackrel{?}{=} 0 \qquad \text{and } \alpha = \left(\frac{\hbar^2}{mk}\right)^{1/4}$$

$$\cancel{km}\left(\frac{\hbar^2}{\cancel{mk}}\right) - \hbar^2 = \hbar^2 - \hbar^2 = 0 \qquad [\text{YES!}]$$

$$\text{so } E_1 = \frac{3\hbar^2}{2m\alpha^2} = \frac{3\hbar^2}{2m}\sqrt{\frac{mk}{\hbar^2}} = \frac{3\hbar}{2}\sqrt{\frac{k}{m}} = \frac{3}{2}\hbar\omega,$$

agreeing with general formula: $E_v = (v+\frac{1}{2})\hbar\omega$

$= \frac{3}{2}\hbar\omega$ when $v = 1$

(!)