Why do catenaries work for the square wheel?

The main idea is that the inverted catenary works for a wheel that is simply an infinite straight line. Given that fact, one just cuts off the catenary where the tangent is at a 45° angle. This means two of them come together to form a right angle, which accepts the square corner perfectly. Here is the flat wheel, with its center considered to be at (0, 0). In polar coordinates this straight line is \( r = -\csc \theta \).

Here are several copies of the straight line as they roll along a catenary. Note that the dots representing the "center" of the straight-line "wheel" stay on a horizontal line.

Now, why does the catenary work for a straight line? Here are the highlights:

1. The polar form of the straight line wheel is \( r = -\csc \theta \).

2. Let \( \theta(x) \) represent the amount of angle the wheel has rolled when its center has horizontal coordinate \( x \).

3. Let \( y = f(x) \) denote the road shape, then \( r(\theta(x)) = -f(x) \).

4. The fact that the arc lengths along the wheel and road must match leads to the differential equation \( \frac{d\theta}{dx} = \frac{1}{r(\theta)} \), with initial conditions \( \theta(0) = -\pi/2 \). Separating the variables and using the cosecant form just given leads to \( x = \int_{-\pi/2}^{\theta} r(\theta) \, d\theta \) or \( x = \int_{-\pi/2}^{\theta} -\csc \theta \, d\theta \).
5. The integral can be evaluated as \( x = -\log(-\tan \frac{\theta}{2}) \).

6. The result of the integration inverts to \( \theta = 2 \arctan(-e^{-x}) \)

7. The road is then \( y = f(x) = -r(\theta(x)) = \csc(2 \arctan(-e^{-x})) \), and this simplifies to \( y = -\frac{e^x + e^{-x}}{2} \), an inverted catenary.