Corrected Proof of Lemma 6.3*

May 8, 2007

Lemma 6.3 (Continuity on \([a,b] \implies \text{Unif. Continuity})\). If \(f\) is continuous over the closed and bounded interval \([a,b]\), then it is uniformly continuous over this interval.

Proof: We assume that \(f\) is not uniformly continuous on \([a,b]\) but that it is continuous at every point of \([a,b]\) and show that this leads to a contradiction.

To say that \(f\) is not uniformly continuous over \([a,b]\) means that there is some \(\epsilon > 0\) for which there is no uniform response, no single \(\delta\) that works at every point \(x\) in \([a,b]\). This, in turn, means that given any \(\delta > 0\), we can always find an \(x \in [a,b]\) and another point \(y \in [a,b]\) such that \(0 < |x - y| < \delta\) but \(|f(x) - f(y)| \geq \epsilon\).

We choose an \(\epsilon > 0\) for which there is no uniform response to \(2\epsilon\) and, for each \(n \in \mathbb{N}\), choose \(x_n, y_n\) in \([a,b]\) such that \(|y_n - x_n| < 1/n\) and \(|f(y_n) - f(x_n)| \geq 2\epsilon\). Let \(x = \lim_{n \to \infty} x_n\), which exists because the sequence of \(x_n\) is bounded. Since \(f\) is continuous at \(x\), there is a response \(\delta > 0\) to \(\epsilon\) at \(x\): \(|y - x| < \delta\) implies that \(|f(y) - f(x)| < \epsilon\). Because \(x\) is the upper limit of the \(x_n\), we can find an \(n > 2/\delta\) for which \(|x - x_n| < \delta/2\). It follows that

\[
|y_n - x| \leq |y_n - x_n| + |x_n - x| < \frac{1}{n} + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

but

\[
|f(y_n) - f(x)| \geq |f(y_n) - f(x_n)| - |f(x) - f(x_n)| > 2\epsilon - \epsilon = \epsilon.
\]

Q.E.D.

*with thanks to Donald G. M. Anderson