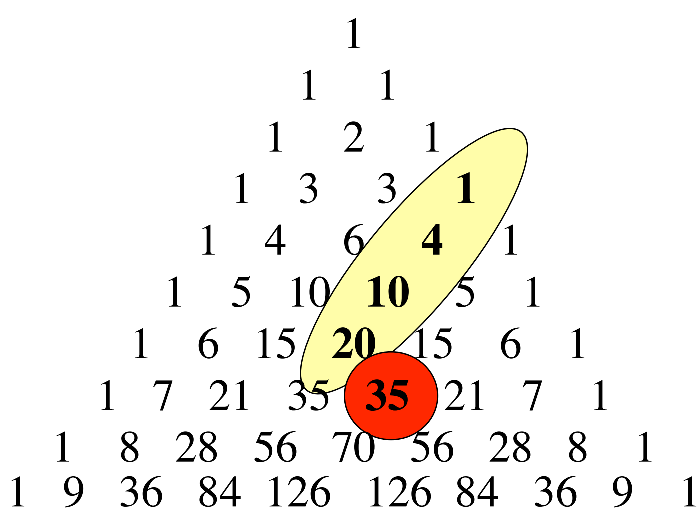
Binomial Coefficients and Sums of nth Powers

Appendix to *A Radical Approach to Real Analysis* 2$^{\text{nd}}$ edition

June 22, 2006

1 Introduction

There is a remarkable property of Pascal's triangle that was independently discovered between the 12th and 14th centuries in India, China, and Europe. If we start along the right-hand edge and come down along any southwest heading diagonal as far as we wish, the sum of the entries we have crossed is equal to the next entry to the southeast.



In this example, we observe that

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} = \binom{7}{4}.$$

In general, we have the formula

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{m-1}{n} = \binom{m}{n+1}. \tag{1}$$

If you think about this little while, you will see that it is a consequence of the fact that each entry is equal to the sum of the two entries that lie above it (see exercise 1).

If we fix a nonnegative integer n, then the binomial coefficient

$$\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$$

is a polynomial of degree n in x that we denote by $P_n(x)$:

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2}x^2 - \frac{1}{2}x, \\
P_3(x) &= \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x, \\
P_4(x) &= \frac{1}{24}x^4 - \frac{1}{4}x^3 + \frac{11}{24}x^2 - \frac{1}{4}x, \\
P_5(x) &= \frac{1}{120}x^5 - \frac{1}{12}x^4 + \frac{7}{24}x^3 - \frac{5}{12}x^2 + \frac{1}{5}x, \\
&\vdots
\end{aligned}$$

Note that if x is a positive integer less than n, then $P_n(x) = 0$. Equation (1) translates into a remarkable insight into this sequence of polynomials:

$$P_n(n) + P_n(n+1) + \cdots + P_n(k-1) = P_n(1) + P_n(2) + \cdots + P_n(k-1) = P_{n+1}(k). \tag{2}$$

We can use this to find sums of arbitrary powers because any polynomial of degree n, including x^n, can be expressed in terms of $P_1(x), P_2(x), \ldots, P_n(x)$. For example:

$$x^4 = 24P_4(x) + 36P_3(x) + 14P_2(x) + P_1(x).$$

It follows that

$$\begin{aligned}
1^4 + 2^4 + 3^4 + \cdots + (k-1)^4 &= \sum_{j=1}^{k-1} \big(24P_4(j) + 36P_3(j) + 14P_2(j) + P_1(j)\big) \\
&= 24\sum_{j=1}^{k-1} P_4(j) + 36\sum_{j=1}^{k-1} P_3(j) + 14\sum_{j=1}^{k-1} P_2(j) + \sum_{j=1}^{k-1} P_1(j) \\
&= 24P_5(k) + 36P_4(k) + 14P_3(k) + P_2(k) \\
&= \frac{24}{120}k^5 + \left(\frac{36}{24} - \frac{24}{12}\right)k^4 + \left(\frac{14}{6} - \frac{36}{4} + \frac{24\cdot 7}{24}\right)k^3 \\
&\quad + \left(\frac{1}{2} - \frac{14}{2} + \frac{36\cdot 11}{24} - \frac{24\cdot 5}{12}\right)k^2 + \left(\frac{-1}{2} + \frac{14}{3} - \frac{36}{4} + \frac{24}{5}\right)k \\
&= \frac{1}{5}k^5 - \frac{1}{2}k^4 + \frac{1}{3}k^3 - \frac{1}{30}k.
\end{aligned}$$

It should be clear that if we know how to expand x^n in terms of our polynomials $P_1(x), P_2(x), \ldots, P_n(x)$, then we can use it to find the formula for $1^n + 2^n + \cdots + (k-1)^n$.

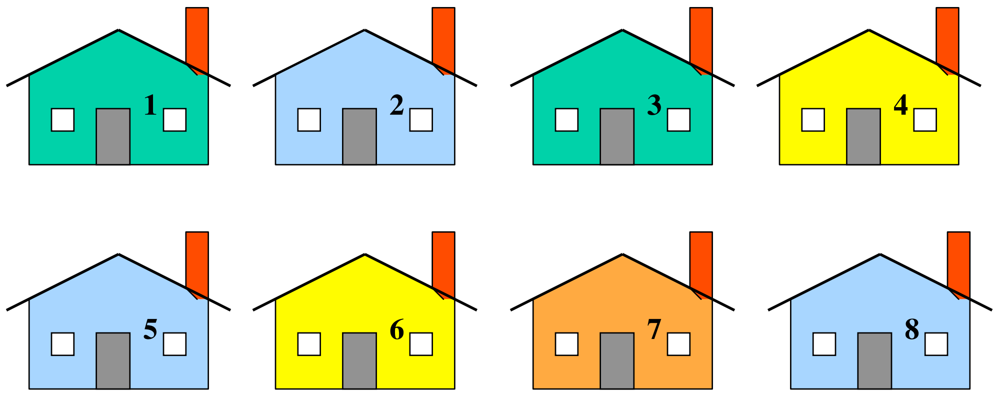


Figure 1: Eight painted houses using exactly four colors.

2 Finding the Coefficients

We shall use a combinatorial or counting argument to find these coefficients. If m and n are positive integers, then m^n counts the number of ways of painting n houses where for each house we have a choice of m colors (see Figure 1). The fact that we have m colors available does not mean that we use all m. We might use only k of our colors. If we do decide to use k colors, then there are $\binom{m}{k}$ choices of which k colors to use. Once we have decide which colors to use, then we want to know how many ways we can paint our n houses using *exactly* k colors. We call this number $HP(n, k)$, the **house-painting number**.

We have shown that

$$m^n = \binom{m}{1} HP(n,1) + \binom{m}{2} HP(n,2) + \binom{m}{3} HP(n,3) + \cdots + \binom{m}{n} HP(n,n). \tag{3}$$

These house-painting numbers are precisely the coefficients that we want. Note that $HP(4, 1) = 1$. If we are using one color, there is only one way to paint four houses. It is also easy to see that $HP(4, 4) = 24$. If we have to use all four colors on four houses, then each house gets a different color, and there are $4! = 24$ ways of assigning the colors. Check for yourself that $HP(4, 2) = 14$ and $HP(4, 3) = 36$.

Notice that we have only established equation (3) when m is a positive integer, but both sides are polynomials in m, so if they agree for all positive integers, then they must agree for all possible values. This equation can be rewritten as

$$x^n = HP(n,1)P_1(x) + HP(n,2)P_2(x) + HP(n,3)P_3(x) + \cdots + HP(n,n)P_n(x). \tag{4}$$

In general, we see that $HP(n, 1) = 1$ and $HP(n, n) = n!$ (see exercise 3). We can start listing these numbers in a triangular arrangements like Pascal's triangle for binomial coefficients:

1				
1	2			
1	6	6		
1	14	36	24	
1	30	150	240	120

This table has a recursion similar to that in Pascal's triangle, but a little more complicated. If we have n houses and must use exactly k colors, we first paint house # 1. There are k choices. We now paint the $n-1$ remaining houses. We have two options for the remaining houses. We can decide that the color used on house # 1 is one we do not want to use again. That leaves us with $HP(n-1,k-1)$ ways of coloring the remaining houses. Or we we can decide that we *do* want to use that color again, leaving us $HP(n-1,k)$ ways of painting the remaining houses. This gives us the recurrsive formula,

$$HP(n,k) = k\left(HP(n-1,k-1) + HP(n-1,k)\right). \tag{5}$$

We add the number above and to the left, to the number above and multiply that sum by the column number.

3 Stirling Numbers

The house-painting numbers are related to a better known collection of numbers known as the **Stirling numbers of the second kind**, $S(n,k)$, by

$$HP(n,k) = k!\,S(n,k).$$

Note that all of the numbers in column k are divisible by $k!$, and it is not hard to see what this must be. If we have k colors for n houses, we can permute the colors and get a different coloring. The Stirling number $S(n,k)$ counts the number of ways of sorting n objects into exactly k non-empty sets. To connect this to the house-painting number, each set of houses gets the same color, and there are $k!$ ways of deciding which color to assign to each set.

The triangle for the Stirling numbers is

1				
1	1			
1	3	1		
1	7	6	1	
1	15	25	10	1

The recursion is

$$\begin{aligned} k!\,S(n,k) &= k\left((k-1)!\,S(n-1,k-1) + k!\,S(n-1,k)\right| \\ S(n,k) &= S(n-1,k-1) + k\,S(n-1,k). \end{aligned} \tag{6}$$

Exercises

1. Prove equation (1) by induction on k.

2. Verify that $HP(4,2) = 14$ and $HP(4,3) = 36$.

3. Explain why $HP(n,1) = 1$ and $HP(n,n) = n!$ for all positive integers n.

4. Use equation 5 to find the values of $HP(6,k)$, $1 \leq k \leq 6$.

5. Use equation 6 to find the values of $S(6,k)$, $1 \leq k \leq 6$.

6. Find the formula for the sum of the 5^{th} powers of the integers from 1 to $k-1$.