The Archimedean Principle

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The **Archimedean principle** states that any two positive distances are commensurable, which means that we can find a finite multiple of the smaller distance that will exceed the larger. This specifically rules out the possibility of infinitesimal distances that are so small that no matter how many of them we take—as long as it is a finite number—we can never get enough to equal or exceed any finite length.

Since we have not yet defined real numbers, it is safest to restate the Archimedean understanding of an infinite series in terms of *rational numbers*, numbers that can be expressed as a ratio of two integers for which the denominator is not zero.

> The value of an infinite series, if it exists, is that number T such given any rational numbers L and M such that $L < T < M$, all of the finite sums from some point on will be strictly contained in the interval between L and M.

We note that positive rational numbers are commensurable (clear denominators and use the fact that positive integers are commensurable) and that between any two rational numbers there always lies a third (the average of two rational numbers is rational).

If we could have numbers that are infinitesimal with respect to the rational numbers, then the Archimedean understanding of infinite series would not work. If the number T is the target values and infinitesimals exist, then there is a positive infinitesimal ι so that for any rational number $M > T$, ι is strictly less than $M - T$. It follows that if L and M are rational numbers for which $L < T < M$, then it is also true that $L < T + \iota < M$. The Archimedean characterization of the target value would not identify it uniquely.

On the other hand, if we accept the Archimedean principle, we can prove that T is uniquely defined. Let us assume that there is another number, U, that also satisfies the definition of a target value. We can assume that $T < U$. Since there is some n for which $n(U - T) > 1$, we know that $0 < 1/n < U - T$, and therefore the smallest multiple of $1/n$ that exceeds T must be strictly smaller than U. This says that there is a rational number of the form m/n that lies strictly between T and U. Thus, if L and M are rational numbers with $L < T < U < M$, then the open interval $(L, m/n)$ contains all terms of the sequence beyond a certain point, but so does $(m/n, M)$, a contradiction.