Assigning Values to Divergent Series

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Cauchy tried to banish the practice of assigning values to series that do not converge.As Daniel Bernoulli showed (see section 2.6), there are traps that are easy to fall into when we attempt to assign values to divergent series. But the fact is, scientists need these values. The work of d'Alembert (see section 2.5) demonstrates the usefulness of such values. A classic example of a divergent series that is nevertheless extremely useful is the common series expansion of the error term in Stirling's formula for n! (see "The size of $n!$" in appendix A.4).

A divergent series cannot give us an arbitarily close approximation to a given value, but it might be able to give us an approximation that is good enough for our purposes. This happens in many areas of science. In the nineteenth century, it occured particularly frequently in astronomy where the values that needed to be calculated could be found to sufficient accuracy by using series that did diverge, but whose divergence was not evident until many terms had been taken. Such series are called asymptotic series.

Much work has been done on divergent series. Ernesto Cesàro and Otto Hölder are among the great mathematicians of the late nineteenth century who worked on them. S. Ramanujan was particularly adept at their manipulation. In 1949, G. H. Hardy published his classic book, *Divergent Series*, on methods of handling and assigning values to such series as well as on demonstrations of their usefulness. But they are not for the novice. It is very easy to fall into error if you are not extremely careful about the assumptions that lie beneath the work you are doing.