Newton's Formula

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We know how Newton discovered his binomial theorem because he described the process in a letter to Leibniz.

Following in Wallis's footsteps, Newton recognized that the key to calculating π was finding a way of evaluating $\pi/4 = \int_0^1 \left(1-x^2\right)^{1/2} dx$. If that exponent were an integer instead of 1/2, life would be easy. Like Wallis, Newton begins by comparing what he has to what he can evaluate. He looks at the expansions of $(1+x)^m$ for integer values of m.

$$\begin{aligned}
(1+x)^0 &= 1+0\cdot x+0\cdot x^2+0\cdot x^3+0\cdot x^4+0\cdot x^5+\cdots, \\
(1+x)^1 &= 1+1\cdot x+0\cdot x^2+0\cdot x^3+0\cdot x^4+0\cdot x^5+\cdots, \\
(1+x)^2 &= 1+2\cdot x+1\cdot x^2+0\cdot x^3+0\cdot x^4+0\cdot x^5+\cdots, \\
(1+x)^3 &= 1+3\cdot x+3\cdot x^2+1\cdot x^3+0\cdot x^4+0\cdot x^5+\cdots, \\
(1+x)^4 &= 1+4\cdot x+6\cdot x^2+4\cdot x^3+1\cdot x^4+0\cdot x^5+\cdots, \\
&\vdots
\end{aligned}$$

He considered a table of the cofficients and tried to guess what coefficients would correspond to an exponent of $m = 1/2$ in

$$(1+x)^m = a_0 + a_1x + a_2x^2 = a_3x^3 + a_4x^4 + a_5x^5 + \cdots.$$

m	x^0	x^1	x^2	x^3	x^4	x^5
0	1	0	0	0	0	0
1/2						
1	1	1	0	0	0	0
3/2						
2	1	2	1	0	0	0
5/2						
3	1	3	3	1	0	0
7/2						
4	1	4	6	4	1	0
9/2						
5	1	5	10	10	5	1

It is easy to guess that the values in the first column are all 1, and the values in the second column must equal m. What about the third column?

Newton would have been very familiar with the sequence 1, 3, 6, 10, , the **triangular numbers**. The jth triangular number is the sum of the integers from 1 to j. It equals $j(j+1)/2$. The exponent m corresponds to the $(m-1)$st triangular number, so the formula to use in the third column is $m(m-1)/2$.

If the values in the first column are constant, the values in the second column increase linearly, and the values in the third column are given by a quadratic formula, then it makes sense to look for a cubic polynomial for the fourth column, a quartic polynomial for the fifth, and so on. Armed with this assumption, it is not difficult to determine what these polynomials must be.

We know that the cubic polynomial in m that fits the coefficients of x^3 must have roots at $m = 0$, 1, and 2. This cubic polynomial must be $cm(m-1)(m-2)$ for some still to be determined constant c. We can find c by using the fact that that this polynomial is 1 when $m = 3$:

$$1 = c \cdot 3(3-1)(3-2) = 6c.$$

This polynomial is $m(m-1)(m-2)/6$.

A similar argument shows us that the polynomial in the next column should be $m(m-1)(m-2)(m-3)/4!$ and the polynomial in the fifth column should be $m(m-1)(m-2)(m-3)(m-4)/5!$.

In general, the column that corresponds to x^k will have zeros at $m = 0, 1, 2, \ldots, k-1$, and a 1 at $m = k$. The corresponding polynomial is

$$\frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}.$$

Since this is defined for any value of m, we can fill in the table:

m	x^0	x^1	x^2	x^3	x^4	x^5
0	1	0	0	0	0	0
1/2	1	1/2	−1/8	1/16	−5/128	7/256
1	1	1	0	0	0	0
3/2	1	3/2	3/8	−1/16	3/128	−3/256
2	1	2	1	0	0	0
5/2	1	5/2	15/8	5/16	−5/128	3/256
3	1	3	3	1	0	0
7/2	1	7/2	35/8	35/16	35/128	−7/256
4	1	4	6	4	1	0
9/2	1	9/2	63/8	105/16	315/128	63/256
5	1	5	10	10	5	1

All of this has been inspired guesswork. Newton did not supply a proof at this time, but he recognized that this enabled him to calculate π with great accuracy, and therefore he was certain that it must be correct. He had discovered the general binomial theorem:

$$(1+x)^m = 1+mx+\frac{m(m-1)}{2!}\,x^2+\frac{m(m-1)(m-2)}{3!}\,x^3+\cdots+\frac{m(m-1)\cdots(m-k+1)}{k!}\,x^k+\cdots. \tag{1}$$

