Newton’s Formula

Appendix to A Radical Approach to Real Analysis 2nd edition
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We know how Newton discovered his binomial theorem because he described the process in a letter to Leibniz.

Following in Wallis’s footsteps, Newton recognized that the key to calculating $\pi$ was finding a way of evaluating $\pi/4 = \int_0^1 (1 - x^2)^{1/2} dx$. If that exponent were an integer instead of $1/2$, life would be easy. Like Wallis, Newton begins by comparing what he has to what he can evaluate. He looks at the expansions of $(1 + x)^m$ for integer values of $m$.

\[
\begin{align*}
(1 + x)^0 &= 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots, \\
(1 + x)^1 &= 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots, \\
(1 + x)^2 &= 1 + 2 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots, \\
(1 + x)^3 &= 1 + 3 \cdot x + 3 \cdot x^2 + 1 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots, \\
(1 + x)^4 &= 1 + 4 \cdot x + 6 \cdot x^2 + 4 \cdot x^3 + 1 \cdot x^4 + 0 \cdot x^5 + \cdots, \\
\vdots
\end{align*}
\]

He considered a table of the coefficients and tried to guess what coefficients would correspond to an exponent of $m = 1/2$ in

\[(1 + x)^m = a_0 + a_1 x + a_2 x^2 = a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots .\]
It is easy to guess that the values in the first column are all 1, and the values in the second column must equal $m$. What about the third column?

Newton would have been very familiar with the sequence 1, 3, 6, 10, , the triangular numbers. The $j$th triangular number is the sum of the integers from 1 to $j$. It equals $j(j+1)/2$. The exponent $m$ corresponds to the $(m - 1)$st triangular number, so the formula to use in the third column is $m(m - 1)/2$.

If the values in the first column are constant, the values in the second column increase linearly, and the values in the third column are given by a quadratic formula, then it makes sense to look for a cubic polynomial for the fourth column, a quartic polynomial for the fifth, and so on. Armed with this assumption, it is not difficult to determine what these polynomials must be.

We know that the cubic polynomial in $m$ that fits the coefficients of $x^3$ must have roots at $m = 0$, 1, and 2. This cubic polynomial must be $cm(m - 1)(m - 2)$ for some still to be determined constant $c$. We can find $c$ by using the fact that that this polynomial is 1 when $m = 3$:

$$1 = c \cdot 3(3 - 1)(3 - 2) = 6c.$$  

This polynomial is $m(m - 1)(m - 2)/6$.  

A similar argument shows us that the polynomial in the next column should be $m(m - 1)(m - 2)(m - 3)/4!$ and the polynomial in the fifth column should be $m(m - 1)(m - 2)(m - 3)(m - 4)/5!$.

In general, the column that corresponds to $x^k$ will have zeros at $m = 0, 1, 2, \ldots, k - 1$, and a 1 at $m = k$. The corresponding polynomial is

$$\frac{m(m - 1)(m - 2) \cdots (m - k + 1)}{k!}.$$  

Since this is defined for any value of $m$, we can fill in the table:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x^0$</th>
<th>$x^1$</th>
<th>$x^2$</th>
<th>$x^3$</th>
<th>$x^4$</th>
<th>$x^5$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>1/2</td>
<td>-1/8</td>
<td>1/16</td>
<td>-5/128</td>
<td>7/256</td>
</tr>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<tr>
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<td>5/16</td>
<td>-5/128</td>
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</tr>
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<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
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<tr>
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</tr>
<tr>
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<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

All of this has been inspired guesswork. Newton did not supply a proof at this time, but he recognized that this enabled him to calculate $\pi$ with great accuracy, and therefore he was certain
that it must be correct. He had discovered the general binomial theorem:

\[ (1+x)^m = 1 + mx + \frac{m(m - 1)}{2!} x^2 + \frac{m(m - 1)(m - 2)}{3!} x^3 + \ldots + \frac{m(m - 1) \cdots (m - k + 1)}{k!} x^k + \ldots. \]  

(1)