Euler's Solution to the Vibrating Drumhead

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One example of the utility of infinite series can be found in Leonhard Euler's analysis of 1759 of the vibrations of a circular drumhead. Euler was led to the differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(\alpha^2 - \frac{\beta^2}{r^2}\right) u = 0, \tag{1}$$

where u (the vertical displacement) is a function of r (the distance from the center of the drum) and α and β are constants depending on the properties of the drumhead. There is no closed form for the solution of this differential equation, but if we assume that the solution can be expressed as a power series,

$$u = r^\lambda + a_1 r^{\lambda+1} + a_2 r^{\lambda+2} + a_3 r^{\lambda+3} + \cdots,$$

then we can solve for λ and the a_i. The derivatives of our power series are

$$\frac{du}{dr} = \lambda r^{\lambda-1} + (\lambda+1)a_1 r^\lambda + (\lambda+2)a_2 r^{\lambda+1} + \cdots, \tag{2}$$

$$\frac{d^2u}{dr^2} = (\lambda-1)\lambda r^{\lambda-2} + (\lambda)(\lambda+1)a_1 r^{\lambda-1} + (\lambda+1)(\lambda+2)a_2 r^\lambda + \cdots \tag{3}$$

Substituting these series into equation (1), we see that:

$$\begin{aligned}
&\left[(\lambda-1)\lambda + \lambda - \beta^2\right] r^{\lambda-2} && (4)\\
&\quad + \left[\lambda(\lambda+1)a_1 + (\lambda+1)a_1 - \beta^2 a_1\right] r^{\lambda-1} \\
&\quad + \left[(\lambda+1)(\lambda+2)a_2 + (\lambda+2)a_2 - \beta^2 a_2 + \alpha^2\right] r^\lambda \\
&\quad + \cdots + \left[(\lambda+j-1)(\lambda+j)a_j + (\lambda+j)a_j - \beta^2 a_j + \alpha^2 a_{j-2}\right] r^{\lambda+j-2} \\
&\quad + \cdots = 0. && (5)
\end{aligned}$$

Each of these coefficients must be zero, and so

$$\lambda = \beta, \tag{6}$$

$$a_1 = 0, \tag{7}$$

$$a_2 = \frac{-\alpha^2}{2(2\beta+2)}, \tag{8}$$

$$a_j = \frac{-\alpha^2}{j(2\beta+j)}\, a_{j-2}, \quad j > 2. \tag{9}$$

It follows that

$$\begin{aligned} u(r) &= r^{\beta}\left[1 - \frac{1}{(\beta+1)}\left(\frac{\alpha r}{2}\right)^2 + \frac{1}{2!\,(\beta+1)(\beta+2)}\left(\frac{\alpha r}{2}\right)^4\right. \\ &\qquad \left. - \frac{1}{3!\,(\beta+1)(\beta+2)(\beta+3)}\left(\frac{\alpha r}{2}\right)^6 + \cdots\right]. \end{aligned} \tag{10}$$

There is no better representation for the solution of this differential equation.