To get out sequence that approaches $\pi$, we begin by splitting off the great integer less than or equal to $\pi$,

floor$\pi$] = $a_1$ = 3,

$$\pi = 3 + r_1, \quad r_1 = 0.14159 \ldots.$$

The remainder, $r_1$, lies between 0 and 1, so its reciprocal is larger than 1:

$$\frac{1}{r_1} = 7.0625133 \ldots.$$

Again, we split off the greatest integer, $a_2$ = 7, and consider the new remainder $r_2 = 0.0625133 \ldots$. We keep doing this, generating a sequence of positive integers:

$$a_1 = 3, \quad a_2 = 7, \quad a_3 = 15, \quad a_4 = 1, \quad a_5 = 292, \quad a_6 = 1, \quad a_7 = 1, \quad a_8 = 1, \quad a_9 = 2, \quad \ldots.$$

If we stop after the $k$th integer, we get a rational approximation to $\pi$,

$$\frac{p_1}{q_1} = 3 = \frac{3}{1},$$

$$\frac{p_2}{q_2} = 3 + \frac{1}{7} = \frac{22}{7},$$

$$\frac{p_3}{q_3} = 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106},$$

$$\frac{p_4}{q_4} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{292}}} = \frac{355}{113},$$

$$\vdots$$

This is called a continued fraction. There is a special notation that makes it easier to write long continued fractions:

$$\frac{p_9}{q_9} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}}}}} = \frac{833719}{265381}.$$
These give the best possible rational approximations for a given limit on the denominator. Specifically, we shall see that if $1 \leq b < q_k$, then no fraction with denominator $b$ can be closer to $\pi$ than $p_k/q_k$.

There is nothing special about $\pi$ in all of this. We could have started with any other irrational number such as $\sqrt{2}$ or $e$. In fact, while the sequence for $\pi$ has no discernible pattern, there are very simple patterns for the integers in the sequences for $\sqrt{2}$ and $e$. As long as we start we with an irrational number, the sequence will never end. The sequence terminates if and only if we start with a rational number. In what follows, we shall prove everything for an arbitrary irrational number that we call $\alpha$.

We begin by defining the sequence:

$$
\begin{align*}
a_1 &= \lfloor \alpha \rfloor, \\
r_1 &= \alpha - a_1, \\
a_{k+1} &= \left\lfloor \frac{1}{r_k} \right\rfloor, \\
r_{k+1} &= \frac{1}{r_k} - a_{k+1}.
\end{align*}
$$

We also define the sequence of rational approximations,

$$
\frac{p_k}{q_k} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_k}}}. 
$$

Notice that if we replace the last $a_k$ by $a_k + r_k$ (an irrational number), we get an expression that exactly equals our original number,

$$
\alpha = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_k + r_k}}}. 
$$

**Proposition 1.** If we define $p_0 = 1$, $p_1 = a_1$, $q_0 = 0$, $q_1 = 1$, then for all $k \geq 1$, we can define

$$
\begin{align*}
p_{k+1} &= p_k + a_{k+1}p_k, \\
q_{k+1} &= q_k + a_{k+1}q_k. 
\end{align*}
$$

Furthermore, we have that

$$
p_{k+1}q_k - p_kq_{k+1} = (-1)^{k+1},
$$

and, therefore, $\gcd(p_k, q_k) = 1$ for all $k \geq 0$.

**Proof.** We prove these equations by induction. When $k = 1$, we have that

$$
p_2 = a_1a_2 + 1 = p_0 + a_2p_1, \quad q_2 = a_2 = q_0 + a_2q_1, \quad p_2q_1 - p_1q_2 = a_1a_2 + 1 - a_1a_2 = 1.
$$

We can turn $p_k/q_k$ into $p_{k+1}/q_{k+1}$ by taking the continued fraction for $p_k/q_k$ and replacing $a_k$ with $a_k + 1/a_{k+1}$. By our induction hypothesis,

$$
p_k = p_{k-2} + a_kp_{k-1}, \quad q_k = q_{k-2} + a_kq_{k-1}.
$$
Therefore,

\[
\frac{p_{k+1}}{q_{k+1}} = \frac{p_{k-2} + (a_k + 1/a_{k+1})p_{k-1}}{q_{k-2} + (a_k + 1/a_{k+1})q_{k-1}} \\
= \frac{p_{k-2} + a_k p_{k-1} + (1/a_{k+1})p_{k-1}}{q_{k-2} + a_k q_{k-1} + (1/a_{k+1})q_{k-1}} \\
= \frac{p_k + p_{k-1}/a_{k+1}}{p_k + p_{k-1}/a_{k+1}} \\
= \frac{p_{k-1} + a_{k+1} p_k}{q_{k-1} + a_{k+1} q_k}.
\]

By our induction hypothesis, \( p_k q_{k-1} - p_{k-1} q_k = (-1)^k \). Using equations (1) and (2), we have that

\[
\frac{p_{k+1} q_k - p_k q_{k+1}}{q_k q_{k+1}} = \frac{(p_{k-1} + a_{k+1} p_k) q_k - p_k (q_{k-1} + a_{k+1} q_k)}{q_k q_{k+1}} \\
= \frac{p_{k-1} q_k + a_{k+1} p_k q_k - p_k q_{k-1} - a_{k+1} p_k q_k}{q_k q_{k+1}} \\
= \frac{p_{k-1} q_k - p_k q_{k-1}}{q_k q_{k+1}} = -(-1)^k.
\]

By equation (3), any common divisor of \( p_k \) and \( q_k \) is a divisor of 1.

Because we also round down to the next integer, \( p_k/q_k \) is less than \( \alpha \) when \( k \) is odd, greater than \( \alpha \) when \( k \) is even. The difference between two successive approximations is

\[
\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{p_{k+1} q_k - p_k q_{k+1}}{q_k q_{k+1}} = \frac{(-1)^{k+1}}{q_k q_{k+1}}.
\]

This tells us that we can write \( p_k/q_k \) as the sum of an alternating series,

\[
\frac{p_k}{q_k} = a_1 + \sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{q_j q_j+1}.
\]

Since the values of \( q_k \) increase, the summands are decreasing and approach 0. This is an alternating series that converges to \( \alpha \). The partial sums alternate larger and smaller than \( \alpha \).

If \( a/b \) lies between \( p_k/q_k \) and \( p_{k+1}/q_{k+1} \), then

\[
\frac{1}{q_k q_{k+1}} > \left| \frac{a}{b} - \frac{p_k}{q_k} \right| = \frac{|aq_k - bp_k|}{b q_k} \geq \frac{1}{b q_k},
\]

and therefore \( b > q_{k+1} \).