Fifth Konhauser Problemfest

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1. A rectangle has sides of lengths 6 and 7. Partition the rectangle into three pieces so as to minimize the largest diameter, and find that diameter. (The *diameter* of a plane region is the largest distance between two points of the region.)

Note: This problem will be scored on a curve; the smller your value, the better your score. Thus you should submit something, for it might be worth full credit. Note that a proof of minimality is not required, but extra credit will be given if you provide one (you must, of course, provide a proof that your value is actually achieved).

2. In the sequence

$$\frac{4}{1} < \frac{25}{4} < \frac{169}{25} < \frac{1156}{169} < \ldots$$

each denominator after the first is equal to the previous numerator, and each numerator is the smallest square consistent with the sequence's being strictly increasing. Find the limit of the sequence, or prove that it increases indefinitely.

3. The lengths of the sides opposite the angles A, B, C of a triangle ABC are a, b, c.

(a) If $(a, b, c) = (2, 3, 4)$, what is wrong with the following argument? The law of cosines gives $\cos A = \frac{7}{8}$ and $\cos C = -\frac{1}{4}$. Therefore,

$$\cos(2C) = 2\cos^2 C - 1 = -\frac{7}{8} = -\cos A = \cos(p - A) = \cos(B + C),$$

so $2C = B + C$ and the triangle is isosceles. Find a correct simple linear relation between the angles.

(b) If $(a, b, c) = (3, 4, 5)$, then the triangle is right-angled and so $A + B = C$. Is there a simple linear relation between the angles when $(a, b, c) = (4, 5, 6)$?

4. If a, b, c are real numbers and $0 < k < 1$, are the roots of the equation $(x + a)(x + b) = k(x + c)^2$ necessarily real?

5. The sequence

$$\ldots, -\frac{1}{4}, -\frac{1}{2}, 0, 1, 1, -1, -3, \ldots$$

is defined by $u_0 = 0$, $u_1 = 1$, and $u_{n+1} = u_n - 2u_{n-1}$.

(a) Show that, for $m \geq 0$, $u_{-m} = -u_m / 2^m$.

(b) Simplify the expression

$$\frac{1}{\sqrt{-7}}\left\{\left(\frac{1+\sqrt{-7}}{2}\right)^{-13} - \left(\frac{1-\sqrt{-7}}{2}\right)^{-13}\right\}.$$

6. The angles A, B, C of a triangle are such that $A \le B \le C \le \frac{\pi}{2}$. Maximize $\sin A \tan B \cos C$.

7. A pole of length L was standing upright against a vertical wall. Amonkey began to ascend the pole. At the same instant, the foot of the pole began to move out from the wall. The monkey adjusted his speed so that the line drawn from him to the foot of the wall was always perpendicaular to the pole; he arrived at the top of the pole at the same time as the top reached the ground. What was the plane area enclosed by the loop in space that the monkey traveled?

8. Let a, b be the roots of $x^2 - 12x + 1$.

(a) Find a quadratic whose polynomial roots are a^4, b^4.

(b) Prove that

$$\sqrt{35} = \frac{10081}{1704} - e, \text{ where } \frac{1}{2\cdot 1704\cdot 10081} < e < \frac{1}{(2\cdot 1704\cdot 10081)-1}.$$

9. A parabola, initially in the position $y^2 = -4ax$, rolls without slipping on the fixed, equal parabola whose equation is $y^2 = 4ax$. What is the locus of the focus of the rolling parabola (initially at $(-a, 0)$)?

10. The last decimal digit of a square belongs to the set $\mathcal{D}=\{0,1,4,5,6,9\}$. Is it true that any finite sequence of decimal digits may appear to the left of the final digit? That is, given any sequence $a_1, a_2, \ldots, a_n$ with $0 \le a_i \le 9$, are there integers whose squares end in $\ldots a_1 a_2 \ldots a_n d$, where d is suitably chosen from $\mathcal{D}$?