Seventh Konhauser Problemfest

Held at St. Olaf College, Feb. 28, 1998
Problems by George Gilbert
First place: Macalester, 85 points out of 100

1. Let $A = (-2, 4)$ and $B = (3, 9)$. The tangent lines to $y = x^2$ at A and B intersect at C. Find the area of $\triangle ABC$.

2. For what $c > 0$ does $\sum_{n=1}^{\infty} \frac{n^n}{(n+c)^{n+c}}$ converge?

3. A rigid wire is formed in the shape of $y = x^2$, $0 \le x \le 1$. The wire is then laid in the first quadrant with one end at the origin and the other on the positive x-axis. Find the y-coordinate of the highest point on the wire.

4. A 3×4 checkerboard may be tiled with four L-shaped tiles of area 3. Determine all positive integers, n, for which it is possible to tile a $3\times n$ checkerboard with n L-shaped tiles of area 3. Justify your answer.

5(a). Prove that $n^2 + 3n + 2$ is **not** a perfect square for any integer n.

(b). Find the largest integer k for which $n^2 + 3n + k$ is not a perfect square for all positive integers n.

6. A point is chosen at random from the uniform distribution on each side of the unit square. What is the probability that the convex quadrilateral, defined by these four poitns, has area greater than $\frac{1}{2}$? (The uniform distribution on an interval means that the probability that a point lies in a subinterval is proportional to the length of the subinterval).

7. Let (a, b) be an open interval of the real line. Show that it is impossible to find polynomials $p(x)$ and $q(x)$ such that $\tan(x) = \frac{p(x)}{q(x)}$ on (a, b).

8. Let $f : \left[0, \frac{1}{2}\right] \to \mathbb{R}$ satisfy $f(0) = 1$ and $f'(x) = f(2x^2)$. Estimate the maximum value of f to within $\frac{1}{4}$.

9. Let $n > 1$ be a positive integer and let $A_1, \ldots, A_n$ be n points in the plane. Let $B_1, \ldots, B_n$ be a rearrangement (permutation) of these points. Prove:

(a) $B_1 B_2 + B_2 B_3 + \ldots + B_{n-1} B_n + B_n B_1 \le \left\lfloor \frac{n}{2} \right\rfloor (A_1 A_2 + A_2 A_3 + \ldots + A_{n-1} A_n + A_n A_1)$.

(b) If $k_n < \left\lfloor \frac{n}{2} \right\rfloor$, then there exist points $A_1, \ldots, A_n$ and a rearrangement $B_1, \ldots, B_n$ such that

$$B_1 B_2 + B_2 B_3 + \ldots + B_{n-1} B_n + B_n B_1 > k_n (A_1 A_2 + A_2 A_3 + \ldots + A_{n-1} A_n + A_n A_1).$$

($P_i P_j$ denotes the distance between two points P_i and P_j, and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

10. Given 1999 real numbers $x_1, \ldots, x_{1999}$ such that $x_1 + \cdots + x_{1999} = 1$, prove that there exists a positive integer n such that

$$\left\lfloor n x_1 + \frac{1}{2} \right\rfloor + \cdots + \left\lfloor n x_{1999} + \frac{1}{2} \right\rfloor = n$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.