1. **Cosine Values.**
   a) Prove that $\cos \frac{\pi}{9}$ is a solution of the equation $8x^3 - 6x = 1$.
   
   b) Prove that $\cos \left( \frac{2\pi}{5} \right) = \frac{-1 + \sqrt{5}}{4}$.

2. **Function Equation.**
   Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $3f(x-1) + f(1-x) = 4x$ for all $x \in \mathbb{R}$.

3. **An Upper Bound.**
   Let $x_1, x_2, \ldots, x_n$ be arbitrary numbers selected from the interval $[0, 1]$ with $n \geq 2$. Show that
   
   $\frac{x_1 + x_2 + \ldots + x_n - x_1x_2 - x_2x_3 - \ldots - x_{n-1}x_n - x_n x_1}{n-1}$
   
   $\leq \left\lfloor \frac{n}{2} \right\rfloor$ where $\left\lfloor \frac{n}{2} \right\rfloor$ represents the greatest integer not exceeding $\frac{n}{2}$.

4. **Multiples of 13?**
   a) Show that $3^{2010} + 5^{2010}$ is divisible by 13.
   
   b) Show that $3^{2004} + 5^{2004}$ is not divisible by 13.

5. **An Integral Inequality.**
   Let $f$ be a continuous function so that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Show that there exists a number $s \in \mathbb{R}$ so that for all $x \in \mathbb{R}$,
   
   $\int_0^x f(s+t) dt \leq x \int_0^1 f(t) dt$.

6. **Serrin’s Integral Formula.**
   Let $f$ be a continuously differentiable function on the interval $[0, 1]$ with $f(0) = 0$. Show that
   
   $\int_0^x \int_0^y \frac{2f(t)f'(t)}{1-t^2} dt du = \int_0^x \frac{(x-t)(1-xt)f(t)^2}{(1-t^2)^2} dt$.
7. **Find the Angle.**
Consider the isosceles triangle ABC below and the given angles (not necessarily drawn to absolutely correct scale). Find (with proof) the measure of angle $\angle DEF$.

![Diagram of isosceles triangle ABC with angles labeled 60°, 20°, 90°, 30°, 30°, 90°.](image)

8. **Sequence of Flips.**
Consider the experiment of flipping a fair coin an infinite number of times with the various flips being independent. Here are two examples from among the infinite number of outcomes of the experiment:

- $H H T T H T H T H T ...$
- $T T H T H T T H T H H H H T H T H T T ...$

where ‘H’ denotes ‘heads’ and ‘T’ denotes ‘tails’. For the first of these outcomes, we notice that $H T T$ (on flips 2, 3, and 4) occurred before $T T H$ (on flips 3, 4, and 5), which in turn occurred before $H T H$ (on flips 8, 9, and 10). For the second outcome, $T T H$ (on flips 1, 2, and 3) occurred before $H T T$ (on flips 5, 6, and 7).

In this experiment,

- a) What is the probability that $T T H$ occurs before $H T T$?
- b) What is the probability that $H T T$ occurs before $H T H$?
- c) What is the probability that $T T H$ occurs before $H T H$?

9. **Guess The Number.**
Ten students tried to guess the 5-digit number that their teacher was thinking of. Nobody succeeded, but each person did get one, and only one, of the 5 digits correct in the right position. What is the teacher's 5-digit number? Here are the 10 guesses:

- 06432, 29751, 94700, 38977, 87036, 43069, 76330, 52025, 61825, 18641

10. **Upper Bound.**
Let $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$ be sequences of positive real numbers such that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0$. Show that for large values of $n$,

$$a_n b_n < \max \left\{ \frac{a_n}{b_n}, e^{\frac{1}{b_n}}, \frac{1}{a_n} \ln \frac{1}{a_n} \right\}$$