The Seventeenth Annual Konhauser Problemfest

Carleton College, February 28, 2009

Problems set by Gregory Galperin, Eastern Illinois University

This contest is held annually in memory of Professor Joseph Konhauser (1924-1992) of Macalester College, who posted nearly 700 Problems of the Week at Macalester over a 25-year period. Joe died in February of 1992, and the contest was started the following year.

INSTRUCTIONS: Each team must hand in all work to be graded at the same time (at the end of the three-hour period). Each problem must be written on a separate page (or pages) and YOUR TEAM NAME SHOULD APPEAR AT THE TOP OF EVERY PAGE. Only one version of each problem will be accepted per team. Calculators of any sort are allowed. Justifications and/or explanations are expected for all problems. All ten problems will be weighted equally, and partial credit will be given for substantial progress toward a solution.

1. **Pencils in boxes**. Each of ten boxes contains a different number of pencils (there is at least one pencil in each box). No two pencils in the same box are of the same color. Prove that one can choose one pencil from each box so that no two are of the same color.

2. **Doubling an integer**. Find the second smallest positive integer with the property that if the last (rightmost) digit of the integer (written, as usual, in base 10) is moved to the front, the new integer formed will be twice the original one. (For example, 1543 does not have this property, because 3154 does not equal $2 \cdot 1543 = 3086$.)

3. **Truncating a polyhedron**. A convex polyhedron $\mathcal{P}$ has 1000 edges. This polyhedron is truncated at each vertex (corner); that is, a small neighborhood of each vertex is cut off by a plane. (If you truncated a cube, you would get a polyhedron with octagonal and triangular faces; the triangular faces would be new, and the octagonal faces would be part of the original square faces of the cube.) For the new polyhedron $\mathcal{Q}$ obtained from $\mathcal{P}$, find the number of vertices, the number of edges, and the number of faces.

4. **Fractional and reciprocal**. The "golden ratio" $\tau = 1.618033989...$ has the property that its fractional part $\{\tau\} = 0.618033989...$ equals its reciprocal. As you might expect, there are other positive real numbers with this property. Find the smallest two such numbers whose difference is between 0.999 and 1.

5. **Mathematical thoughts**. Mathematicians **A**, **B**, and **C** all know that they have positive integers a, b, and c on their foreheads and that one of those integers equals the sum of the two others. Of course, they see each other's integers but not their own. Not surprisingly, after some thought **A** announces that she can't tell what her integer is, then **B** announces the same, then **C** announces the same. But then **A** announces that she now knows her integer, and that it is $a = 50$. What were the other integers, b and c?

6. **Crossing diagonals**. A line passes through the interior of a convex n-gon but does not pass through any of the vertices; inside the n-gon, the line intersects exactly m of the diagonals.
(a) Suppose $m = 2009$. Show that there are exactly three possibilities for n, and find those values of n.
(b) Find the smallest value of m for which there are exactly fourteen possibilities for n.

7. **Halving an "L"**. A figure in the plane is in the shape of an "**L**" formed by two rectangles: a short and fat vertical rectangle and a long and narrow horizontal rectangle, whose left-hand vertical side is also the bottom of the right-hand side of the vertical rectangle. Using only a straightedge, construct a line that dissects the "**L**" shape into two regions of equal area. (Your construction should be independent of the dimensions of the rectangles.)

8. **Pi in your integral**. Evaluate the following definite integral:

$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^\pi)}\,.$$

9. **Venn diagrams**. A "Venn diagram" for three sets is drawn using convex shapes in the plane, so as to form seven bounded, connected regions and one unbounded one. Is it possible that the areas of the seven bounded regions are all equal:
(a) if the three sets are bounded by congruent polygons?
(b) if the three sets are bounded by circles?

10. **A limit of integrals**. Find the limit:

$$\lim_{n\to\infty} \int_0^1 \sqrt{1+n^2x^{2n}}\,dx\,.$$