Third Konhauser Problemfest

St. Olaf College Feb. 25, 1995

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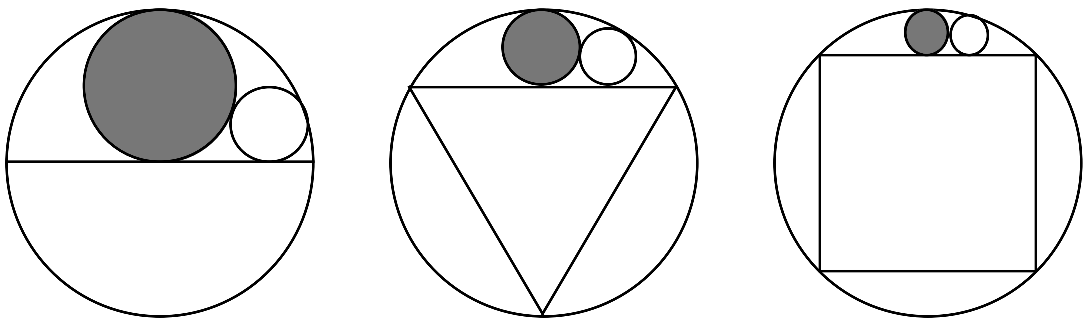
Calculators of any sort are allowed. Justifications are expected, not just the statement of an answer. Partial credit will be given for progress toward a solution or the answer to part of a question.

1. Circular Surprises In the diagram, the large circle has radius one, the inscribed figures are a diameter, an equilateral triangle, and a square, the shaded circle is tangent to the top of the circle, and the other tangencies are as they appear. Determine the radii of the shaded and unshaded circles.



2. A Vintage Year The year 1979 was unusual in that it results from stringing together distinct 2-digit primes, namely 19, 97, and 79. The next time this happens is in 2311, which comes from 23, 31, and 11. When will this happen for the last time? Remember: the primes must be distinct.

3. A Circular Committee A committee with 1,995 members sits around a circular table. Every hour there is a vote and each member must vote either YES or NO. Everyone votes his or her conscience on the first round, but after that the following rule is obeyed: On the nth vote, if a person's vote is the same as *at least one* of the two votes of adjacent committee members, the member votes the same way on the $(n+1)$st round as on the nth. Otherwise the person's $(n+1)$st vote is the opposite of his or her nth. Prove that, regardless of the first-round votes, there will come a time after which no one's votes will change.

4. Complex Relationships Let a, b, c, d denote complex numbers. TRUE or FALSE:

(a). If $a + b = 0$ and $|a| = |b|$, then $a^2 = b^2$.

(b). If $a + b + c = 0$ and $|a| = |b| = |c|$, then $a^3 = b^3 = c^3$.

(c). If $a + b + c + d = 0$ and $|a| = |b| = |c| = |d|$, then $a^4 = b^4 = c^4 = d^4$.

5. The Middle of a Moving Line Suppose that L and M are two nonintersecting lines in 3-space that are perpendicular to each other. A line segment PQ of fixed length moves so that P is on L and and Q is on M. What is the locus of the midpoint of PQ?

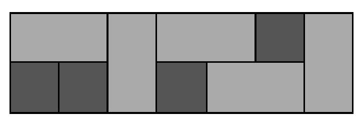
NOTE: A line L in 3-space is perpendicular to another line M if M is contained in a plane perpendicular to L.

6. Visible People Suppose n people, all having distinct heights, are standing in a single-file line. Call a person "visible" if he or she is taller than anyone in front of him or her (and so is visible to a person looking at the line from the front). Assuming a random distribution of the people into the lines, how large must n be in order that the expected number of visible people is 10?

7. Find the Pattern The squares of an infinite chessboard are numbered as in the sketch. The number 0 is placed in the lower left-hand corner; each remaining square is numbered with the smallest nonnegative integer that does not already appear to the left of it in the same row or below it in the same column. If the first row (column) is called the zeroth row (column), which number will appear in the 939th row and 1120th column?

5	4	7	6	1	0	
4	5	6	7	0	1	
3	2	1	0	7	6	
2	3	0	1	6	7	
1	0	3	2	5	4	
0	1	2	3	4	5	

8. Count the Tilings The diagram shows a tiling of a 2×7 rectangle with 1×1 and 1×2 tiles (singletons and dominoes; dominoes may be placed horizontally or vertically). How many such tilings of a 2×7 grid are there?



9. 11111111111111... Prove that there are infinitely many integers n for which the base-10 number obtained by stringing together n 1s is divisible by n. For example, 111 is divisible by 3.

10. Three Rising Vectors A *rising vector* in the plane is one whose vertical component is nonnegative. The sum of two rising vectors of length one can be very short. Prove that the sum of three rising vectors, all having length one, cannot have length less than one.