

Sixth Konhauser Problemfest

Held at University of St. Thomas, Feb. 28, 1998

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First place: Macalester, 96 points out of 100

1. Find all triples of integers (a, b, n) satisfying $\frac{a}{b} + \frac{b}{a} = \frac{a+b}{n}$.

2. Find c such that the parabolas $y = -x^2$ and $y = 2x^2 + c$ have two common tangent lines that meet at a right angle.

3. Find the area of the following intersection of planar sets:

$$\{(x, y) \mid x^2 + (y - 2\sqrt{3})^2 \leq 36\} \cap \{(x, y) \mid (x - 3)^2 + (y + \sqrt{3})^2 \leq 36\} \\ \cap \{(x, y) \mid (x + 3)^2 + (y + \sqrt{3})^2 \leq 36\}$$

4. Classify the triples (A, B, C) according to whether the intersection of the plane $z = Ax + By + C$ with the paraboloid $z = x^2 + y^2$ is empty, is a single point, or is an ellipse (a circle being a special case of an ellipse). In the case of an ellipse, find its area.

5. Find the smallest positive integer that has exactly 12 distinct positive integral divisors and whose prime divisors all lie between 10 and 20.

6. A permutation matrix is a square matrix all of whose entries are 0s and 1s with there being exactly one 1 in each row and in each column. Let b_n denote the number of n -by- n permutation matrices having the additional property that the row i and the column j of every position in which a 1 appears satisfy $i-1 \leq j \leq i+2$. Find a recursive description of the sequence $(b_1, b_2, b_3, b_4, \dots)$ and use this recursive definition to calculate b_6 .

7. Prove that $\lim_{n \rightarrow \infty} \left[2\sqrt{n} - \sum_{k=1}^n \frac{1}{\sqrt{k}} \right]$ exists and that the value of this limit is between $1 + 2^{-3/2}$ and $1 + 2^{-1}$.

8. Let $x, y,$ and z denote lengths of the three medians of an arbitrary triangle. Prove that $x < y + z$. (Recall that the median of a triangle is a line segment, one of whose endpoints is a vertex and the other of which is the midpoint of the opposite side. The major standard theorem about medians is that they meet in a common point that on each median is twice as far from its vertex endpoint as it is from its other endpoint.)

9. For $x \geq 1$, let $f_0(x) = x$ and for $n > 0$, $f_n(x) = \max\{1, \ln(f_{n-1}(x))\}$. Set $g(x) = \frac{1}{f_0(x) f_1(x) f_2(x) \dots}$. Decide whether $\int_1^{\infty} g(x) dx$ converges or diverges.

10. Numbers X_1, X_2, X_3, \dots are chosen independently at random from the interval $(0, 1)$ according to the uniform distribution. Define $K = \infty$ if sequence $\{X_1, X_2, X_3, \dots\}$ is strictly monotone increasing and $K = \min \{n : X_n > X_{n+1}\}$ otherwise. Calculate the probability that $K = k$ for $k = 1, 2, 3, \dots$ and show that the sum of these probabilities is equal to 1.