Konhauser Problemfest 2003 St Olaf
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1. Place six distinct positive integers on a cube, one per face. Form at each corner the product of the three numbers on the faces at that corner and add the eight such products together. Is it possible that this total is 385?

2. Tile the first quadrant of the plane with 1x1 squares, and into each square place the quantity 2^{-x-y}, where (x, y) are the coordinates of the lower-left corner of that square. Now tile the first quadrant of the plane with 1x2 dominoes, and let S be the infinite sum of the products of the two numbers found inside each domino. Find, with proof, all possible values of S. Each value should be given as a fraction in lowest terms. (Hint: For |x| < 1, the power series $1 + 2x + 3x^2 + 4x^3 + \ldots$ converges to a reasonably simple rational function.)

3. A round-robin Clobber tournament was played with n ≥ 4 players. (In a round-robin tournament, each player plays once against each other player, with each game having a winner.) A "champion" is a player a_1 so that there exists an ordering a_1, a_2, a_3, ..., a_n of the players (called a "champion ordering") so that a_i beats a_{i+1} for 1 ≤ i < n. Here is a fact that you could prove if you had more time: Given any sequence of distinct players a_1, a_2, ..., a_k, k ≤ n, with a_i beating a_{i+1} for $1 \le i \le k-1$, there exists a champion ordering of all the players that preserves the ordering of the given sequence of players.

Using this fact (or not) prove that the tournament must have had at least one champion, but that it could not have had exactly two champions.

4. Write 10! as a sum of the smallest possible number of perfect squares, and prove that fewer squares will not work.

5. Let α = p/q and β = r/s be two rational numbers, with p, q, r, and s > 0 and $ps - qr = 1$. For all integers i ≥ 0, let a_i be the point (Floor[i α], 0) and b_i be the point (Floor[i β], 1). How many of the line segments $a_i\ b_i$ will be vertical?

6. Let $p(x) = x^5 - x^3 - x - 3$ and $q(x) = x^3 + x^2 + x + 1$, and let r_1, r_2, r_3, r_4, and r_5 be the roots of p. Find $q(r_1)\,q(r_2)\,q(r_3)\,q(r_4)\,q(r_5)$.

7. An unbalanced penny and an unbalanced quarter, with probabilities of heads p for the penny and q for the quarter, are tossed together over and over. The probability that the penny shows heads (strictly) before the quarter does is 3/5, and the number of tosses required for both coins to show heads simultaneously has expected value exactly 4. Find the values of p and q.

8. Consider the surface S in 3-space consisting of those points (x, y, z) that satisfy the equation $x^2 + y^2 - 2z^2 = 1$. If P is any point of S, there are exactly two straight lines that are contained in S and contain the point P. Find the locus of those points P of S with the property that the two straight lines associated to P as above are perpendicular.

9. What is the probabiliy that a randomly selected 3x3 matrix of zeros and ones will have nonzero determinant?

10. Determine whether the quantity

$$s = \sqrt{1^2 + \sqrt{2^2 + \sqrt{3^2 + \sqrt{\cdots + \sqrt{n^2 + \sqrt{\cdots}}}}}}$$

is finite or infinite.