Sixteenth Annual Konhauser Problemfest

Macalester College **Feb. 23, 2008**

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Calculators of any sort are allowed. Justifications are expected, not just the statement of an answer. Partial credit will be given for progress toward a solution or the answer to part of a question.

1. Leave a Tip I was at a restaurant the other day. The bill came, and I wanted to give the waiter a whole number of dollars, with the difference between what I give him and the bill being the tip. I always like to tip between 15% and 20% of the bill. But if I gave him a certain number of dollars, the tip would have been **less** than 15% of the bill, and if instead I gave him one dollar more, the tip would have been **more** than 20% of the bill. What was the largest possible amount of the bill?

1. Solution The answer is \$19.16. We work in cents. The two main conditions, where B is the bill and x is the number of dollars, are $100\,x - B < \frac{15}{100}\,B$ and $100\,(x+1) - B > \frac{20}{100}\,B$. Reducing fractions: $100\,x < \frac{23}{20}\,B$ and $100\,(x+1) > \frac{6}{5}\,B$ or $\frac{2000}{23}\,x < B < \frac{250}{3}\,(x+1)$. The implied inequality on x yields $\frac{8}{23}\,x < \frac{1}{3}\,(x+1)$, or $\frac{24}{23}\,x < x+1$ or $\frac{x}{23} < 1$. So x must be less than 23, and we try 22. B must then lie between $\frac{2000\times 22}{23}$ and $\frac{250\times 23}{3}$. These are 1913.04 and 1916.67, so the amount of the bill was \$19.16. Such problems can be completely solved by computer, as the following *Mathematica* code demonstrates,

```
Maximize[
 {B, {100 x > B, 100 x - B < 15 B/100, 100 (x + 1) - B > B/5, x ≥ 1, B ≥ 1, {B, x} ∈ Integers}},
 {B, x}]

{1916, {B → 1916, x → 22}}
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2. Dedicated to Ramanujan The number 1729 has the property that it is divisible by the sum of its digits. In fact, 1728 has this property as well. Find three consecutive numbers larger than 1729 such that each is divisible by the sum of its decimal digits.

2. Solution While a computer search will yield more interesting examples, the simplest approach to the problem is to use 10010, 10011, 10012. The first is divisible by 2, the second is divisible by 3, and the third is divisible by 4. The smallest sequence that works (ignoring one-digit numbers) is 110, 111, 112. Here is a table of the smallest consecutive sequences with this property of length 1, 2, 3, 4, 5, 6 (again, ignoring one-digit numbers).

10
20, 21
110, 111, 112
510, 511, 512, 513
131052, 131053, 131054, 131055, 131056
10000095, 10000096, 10000097, 10000098, 10000099, 10000100

It is not known if there is a consecutive sequence of length 7 (excluding the starting values 1, 2, 3, or 4).

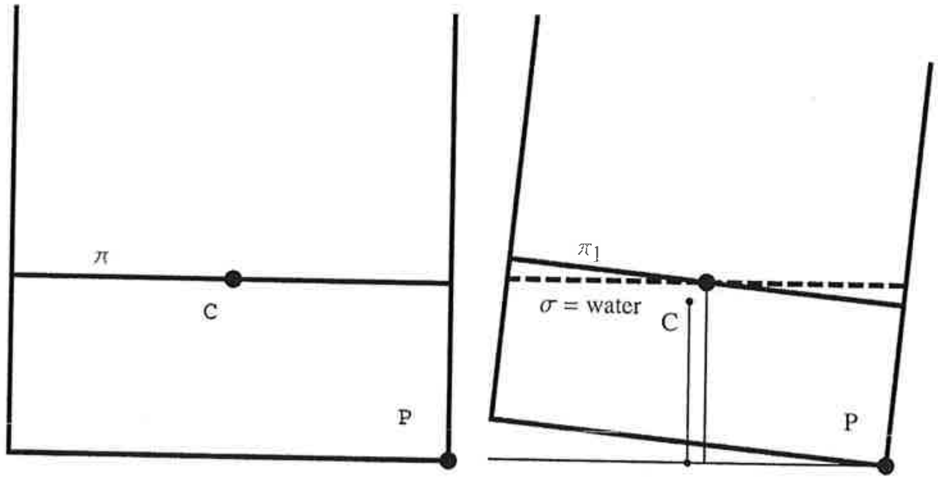
3. Intersection Chance Let p and q be two positive real numbers less than 1 and consider a square of side 1 whose sides are horizontal and vertical. A horizontal line of length p is drawn at random inside the square (the line cannot stick outside the square). Independently, a vertical line of length q is drawn at random inside the square. What is the probability that the two lines intersect?

3. Solution Think of the vertical segment as fixed and the horizontal segment varying. If the horizontal segment is extended to span the square, the probability that the vertical segment intersects the extension is q. Similarly, if the vertical segment is extended to span the square, then the probability that the horizontal segment intersects the extension is p. Since the two segments intersect if and only if each intersects the extension of the other, and since these probabilities are independent, the required probability is $p\,q$.

4. Tip Your Glass A cylindrical glass about one-third filled with water is sitting on a level table. It is then tipped slightly so that its circular base touches the table at one point. Has the distance of the surface of the water from the table increased, decreased, or stayed the same?

4. Solution Let P be the point where the glass forever touches the table. Let Q be the opposite end of the bottom of the glass from P, so PQ is a diameter. Let AB be the diameter that is perpendicular to this, but elevated so that it lies on the surface of the water, pre-tipping. Let C be the midpoint of AB. The point C is on the surface of the water. It is also useful to name the planes. Let π be the plane that is the original surface of the water (so π is horizontal); let π_1 be the location of π after rotation, a tilted plane. Let σ be the horizontal plane that represents the new surface of the water. Let A_1, B_1, and C_1 be the locations of A, B, C, resp. after the tipping through an angle ρ. Remember that ρ is only a small angle as the tipping was "slight".

Now, part of π_1 lies lower than σ and part lies higher. The water that used to be in the region cut off by these two planes, and the cylinder, and is higher than σ has moved into the region that is between these two planes and lower than σ. But these two regions are congruent, by symmetry. It follows that $A_1 B_1$, which lies on π_1, also lies on σ. Therefore C_1 lies on σ and so the height of σ is the height of C_1. But it is clear that C_1 is higher than C, since the rotation made the line PC more vertical as it transformed it to PC_1. This entire argument can be carried out in a 2-dimensional way by looking at cross-sections in vertical planes parallel to the vertical plane through PQ.



5. Old Idaho Usual Here Let w be any n-letter word ($n \geq 1$) containing at most 10 different letters, like *KONHAUSER, PROBLEMFEST, OLDIDAHO*, or *ZZZZZZZZZZZZZZZZZ*. Prove that you can replace the letters of w by decimal digits (different letters replaced by different digits) so that the resulting n-digit number is a multiple of 3.

5. Solution Use the well known fact that a number is divisible by 3 iff the sum of its digits is divisible by 3 (which follows from $10^i \equiv 1 \pmod 3$). Let the 10 letters be A_i, B_i, and C_i, where each A_i has a multiplicity that is divisible by 3, each B_i has a multiplicity of the form $3k+1$, and each C_i has a multiplicity of the form $3k+2$. Let a be the number of A_i, b the number of B_i, and c the number of C_i. We know that $0 \leq b + c \leq 10$. Look for 1, 2, or 3 pairs among the B_i and replace them with digits 1 and 2, and 4 and 5 for the second pair, and 7 and 8 for the third pair. Then look for pairs of the C_i and replace them with digits in any of the still-available pairs from (1, 2), (4, 5), (7, 8). These substitutions take care of $B_i \cup C_i$ except possibly four letters (since we used three pairs) and we can use 0, 3, 6, 9 for them. The letters A_i can be assigned the remaining digits in any order.

It is natural to ask: Which divisors d can be forced in this way (call them *attainable*)? It is easy to see that 2, 5, and 10 are attainable. And 4 and 8 are easily handled as well. The $d = 9$ case can be done by a variation of the $d = 3$ case. The first hard case is $d = 7$; Stan Wagon, while preparing this contest, proved that $d = 7$ is not attainable. An example (this one was found by Robert Israel) is OLDIDAHOUSUALHERE which uses all 10 letters. Work out the mod-7 value of this and you will see it is $3 \cdot (1+2+3+4+5+6+7+8+9) = 3 \cdot 45 \equiv 2 \pmod 7$. Israel and Wagon then found a complete characterization of the attainable integers: they are exactly the set of divisors of 18, 24, 45, 50, 60, or 80. To be explicit, there are exactly 22 attainable integers: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 40, 45, 50, 60, 80. Details on request.

6. Hilbert College In Hilbert College there are an infinite number of lockers, numbered by the natural numbers: 1, 2, 3, … . Each locker is occupied by exactly one student. College President Cantor decides to rearrange the students so that the lockers are all still occupied by the same set of students, one to a locker, but in some other order. (Some students might not change lockers.) It turns out that the locker numbers of infinitely many students end up higher than before. Show that there are also infinitely many students whose locker numbers are lower than before.

6. Solution Suppose that an infinite number of students move to a higher locker number, but that only a finite number move to a lower number locker. There is therefore a locker number N such that no student in a locker greater than N moves to a lower number locker.

Consider lockers 1 to N and note that no student with a locker beyond N moves into one of these. Therefore the rearrangement of students who start with lockers 1 to N is a permutation of the numbers from 1 to N. So the students in lockers $N+1$, $N+2$, $N+3$, ... must be permuted and so there is a smallest numbered locker $M > N$ from which a student moves. M cannot be filled from below, or it would not be smallest. M cannot be filled from above, because it is beyond the maximum N. This is a contradiction.

7. Integers as Dot Products Let $P(k)$ be the assertion: Every integer is a dot product of two arithmetic progressions of integers, each of length k. So, for example, $P(3)$ is true if and only if every integer n has the form $a_1 b_1 + a_2 b_2 + a_3 b_3$, where $\{a_i\}$ and $\{b_i\}$ are arithmetic progressions. Determine the truth or falsity of each of $P(3)$, $P(4)$, $P(5)$, and $P(6)$.

Note: An *arithmetic progression* refers to any sequence of the form a, $a+b$, $a+2b$, $a+3b$, ..., where a and b are arbitrary real numbers.

7. Solution Only $P(3)$ and $P(6)$ are true. For those two we can use the progressions below:

$a_1 b_1 + a_2 b_2 + a_3 b_3 = n$ is solved by $\{0, 1, 2\} \cdot \{2n, n, 0\}$.

$a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + a_5 b_5 + a_6 b_6 = n$ is solved by $\{-3, -2, -1, 0, 1, 2\} \cdot \{3n, 4n, 5n, 6n, 7n, 8n\}$.

For $P(4)$, suppose the first AP is $\{a, a+d, a+2d, a+3d\}$ and the second $\{b, b+c, b+2c, b+3c\}$. The dot product expands to $4ab + 6ac + 6bd + 14cd$, which is even, and so one cannot represent any odd integer. Similar, when $n = 5$ the dot product expands to $5ab + 10ac + 10bd + 30cd$, so that only numbers divisible by 5 are representable.

Further analysis along these lines shows that the only true cases are P(1), P(2), P(3), and P(6). Here is a proof, with *Mathematica*'s help for algebra.

$$\texttt{expr} = \sum_{i=0}^{n-1} (b + i\,c)\,(a + i\,d);$$

```
coeffs = Table[Simplify[Coefficient[expr, t]], {t, {a b, a c, b d, c d}}]
```

$$\left\{n, \frac{1}{2}(-1+n)\,n, \frac{1}{2}(-1+n)\,n, \frac{1}{6}n\left(1-3n+2n^2\right)\right\}$$

We can identify these terms as n, $\binom{n}{2}$, and $2\binom{n}{3}+\binom{n}{2}$. Here is a check.

```
Expand[{n, Binomial[n, 2], Binomial[n, 2], 2 Binomial[n, 3] + Binomial[n, 2]} -
  coeffs]
```

```
{0, 0, 0, 0}
```

So we are done once we identify the cases in which there is a nontrivial common divisor to n, $\binom{n}{2}$, and $2\binom{n}{3}+\binom{n}{2}$. But this is the same as asking for a nontrivial GCD for n, $\binom{n}{2}$, and $\binom{n}{3}$. Breaking it down mod 6, we find that for the 6 cases, starting with 0 mod 6, a common divisor is $n/6$, n, $n/2$, $n/3$, $n/2$, n. Here is confirmation

```
Binomial[6 k , {1, 2, 3}]
------------------------- // Simplify
           k
Binomial[6 k + 1 , {1, 2, 3}]
----------------------------- // Simplify
           6 k + 1
Binomial[6 k + 2 , {1, 2, 3}]
----------------------------- // Simplify
           3 k + 1
Binomial[6 k + 3 , {1, 2, 3}]
----------------------------- // Simplify
           2 k + 1
Binomial[6 k + 4 , {1, 2, 3}]
----------------------------- // Simplify
           3 k + 2
Binomial[6 k + 5, {1, 2, 3}]
---------------------------- // Simplify
           6 k + 5

{6, -3 + 18 k, (-2 + 6 k) (-1 + 6 k)}

{1, 3 k, k (-1 + 6 k)}

{2, 1 + 6 k, 2 k (1 + 6 k)}

{3, 3 + 9 k, (1 + 3 k) (1 + 6 k)}

{2, 3 + 6 k, 2 (1 + 2 k) (1 + 3 k)}

{1, 2 + 3 k, 2 + 7 k + 6 k^2}
```

Quicker way to confirm:

```
{Simplify[Divisible[Binomial[6 k , {1, 2, 3}], k], k ∈ Integers],
 Simplify[Divisible[Binomial[6 k + 1 , {1, 2, 3}], 6 k + 1], k ∈ Integers],
 Simplify[Divisible[Binomial[6 k + 2, {1, 2, 3}], 3 k + 1], k ∈ Integers],
 Simplify[Divisible[Binomial[6 k + 3, {1, 2, 3}], 2 k + 1], k ∈ Integers],
 Simplify[Divisible[Binomial[6 k + 4, {1, 2, 3}], 3 k + 2], k ∈ Integers],
 Simplify[Divisible[Binomial[6 k + 5, {1, 2, 3}], 6 k + 5], k ∈ Integers]}

{{True, True, True}, {True, True, True}, {True, True, True},
 {True, True, True}, {True, True, True}, {True, True, True}}
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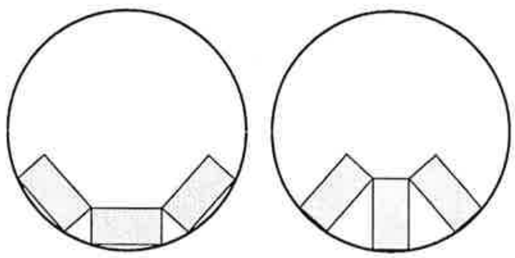
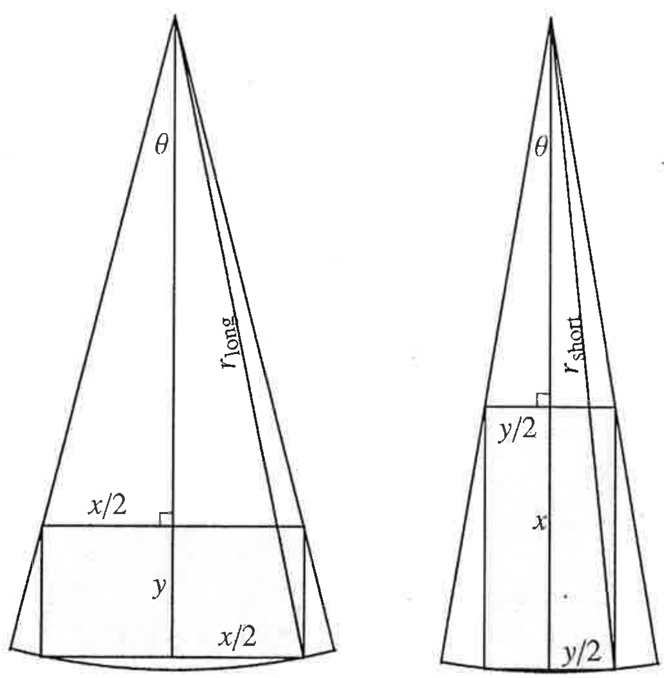
8. Puzzle of the Round Table A cafeteria at a large university has round tables (of every possible radius) and rectangular trays (nonsquare and all the same size). Diners place their trays of food on the table so that two adjacent corners of each tray are on the edge of the table, and with either the short or long sides of the trays pointing toward the center of the table. Moreover, at the same table everybody aligns their trays the same way. Suppose n mathematics students come in to eat together. How should they align their trays so that the table needed is as small as possible?



8. Solution Let the trays have dimensions x and y, with $x > y$. Let r_{short} be the radius of the table needed when the short sides are against the round part, and r_{long} the radius for the other choice. Each tray is inscribed in a circular sector with central angle $2\pi/n$, Let $\theta = \pi/n$. Then (see diagrams):

$r_{\text{long}}^2 = \left(\frac{x}{2}\right)^2 + \left(y + \frac{x}{2}\cot\theta\right)^2$ and $r_{\text{short}}^2 = \left(\frac{y}{2}\right)^2 + \left(x + \frac{y}{2}\cot\theta\right)^2$. Subtraction yields:

$r_{\text{long}}^2 - r_{\text{short}}^2 = \frac{x^2-y^2}{4}\left(\cot^2\theta - 3\right)$. So $r_{\text{long}} > r_{\text{short}}$ iff $\cot^2\theta > 3$ iff $\theta < \frac{\pi}{6}$ iff $n > 6$. Therefore to obtain the smallest table place the short side against the circle if $n > 6$, and place the long side against the circle if $n < 6$. If $n = 6$, there is no difference.



9. Find a Transversal Let X be the set of three numbers $\left\{0, \frac{4}{11}, 1\right\}$. Find a set of numbers that intersects every translation of X in exactly one number. Note: A translation of X by t is the set $\left\{t, \frac{4}{11}+t, 1+t\right\}$.

9. Solution Use the set of point of the form $x+\frac{3k}{11}$, where $0 \le x < 1$ and $k \in \mathbb{Z}$. The main idea is to observe that, in $\mathbb{Z}$, the set $\{0, 1, 2\}$ has the "transversal" consisting of all multiples of 3. That is, any translation $\{0, 1, 2\} + T$ $(T \in \mathbb{Z})$ contains one and only one multiple of 3. Of course, the same is true for the set $\{0, 4, 11\}$ since congruence mod 3 is all that matters. Having a transversal for $\{0, 4, 11\}$ in $\mathbb{Z}$ easily yields one for the given set in $\mathbb{R}$.

Alternative: Easier with $\left\{0, \frac{1}{11}, 1\right\}$. Easier still with $\left\{0, \frac{1}{5}, 1\right\}$. Trivial with $\left\{0, \frac{1}{2}, 1\right\}$.

In fact, such transversals exist for a set $\left\{0, \frac{m}{n}, 1\right\}$ if and only if the integers m and n, when reduced mod 3, are 1 and 2, in either order (see Gao, Miller, and Weiss, Steinhaus sets and Jackson sets, to appear, Contemporary Mathematics)

10. Sort the Cards You are given three cards. Each card has three nonnegative real numbers written on it from top to bottom, with sum 1. Show that you can put the three cards in some order so that the sum of the first number on the first card, the second number on the second card, and the third number on the third card lies in the closed interval $\left[\frac{1}{2}, \frac{3}{2}\right]$.

10. Solution View the cards as follows:

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}$$

In the order shown, the sum we seek is $a_1 + b_2 + c_3$. If we switch the first two cards the sum becomes $a_2 + b_1 + c_3$. We will here identify one of these sums with the permutation that yields it. We wish to prove that at least one of the six permutations of the cards leads to a sum between $\frac{1}{2}$ and $\frac{3}{2}$, inclusive. We will do this by showing that (1) at most two of the permutations have a sum strictly greater than $\frac{3}{2}$, and (2) at most three of the permutations have a sum strictly less than $\frac{1}{2}$. That leaves at least one in the form we want.

Proof of (1) If not we have three permutations with each sum greater than $\frac{3}{2}$. But two of the permutations must be disjoint, since two of them must lie in the first row below, or in the second row below.

$a_1 + b_2 + c_3$	$b_1 + c_2 + a_3$	$c_1 + a_2 + b_3$
$a_1 + c_2 + b_3$	$b_1 + a_2 + c_3$	$c_1 + b_2 + a_3$

These two disjoint permutations have a total 6-number sum that is 3, so at least one of the two sums is in $\left[0, \frac{3}{2}\right]$, contradiction.

Proof of (2) Consider any four permutations.

Claim. They must use all the entries of at least two of the cards.
Proof of claim. There are three entries on each card, so some entry on the first card must appear twice. Relabeling, we may assume this entry is a_1, and it follows that two of the four permutations are $a_1 + b_2 + c_3$ and $a_1 + c_2 + b_3$ (column 1 in the array displayed above). If the four permutations do not include all of card 1, then the other two must be the second column in the array displayed above (in which case all of cards 2 and 3 are represented), or the third column (in which case again all of cards 2 and 3 are represented). And if the four permutations do include all of card 1, then each entry of the last two columns will provide either a_2 or a_3, which is enough to get card 2 or card 3.

Now if S is the sum of the four permutations, S must be at least 2, by the Claim. Therefore it cannot be that all four permutations have sums strictly less than $\frac{1}{2}$, completing the proof.

For a more detailed analysis of this problem, and various extensions and open questions, see "A New Proof and Extension of Problem 2620" by Eric Lenza and Bill Sands, *Crux Mathematicorum* 31 (2005) 319-326.