2010 Konhauser Problem Fest
Solutions

Problems written by Răzvan Gelca

1. Find all complex numbers z that satisfy the equation

$$|3z+1| + |2z-3| + |z-2| = 6.$$

Solution: The triangle inequality implies

$$\begin{aligned}|3z+1| + |2z-3| + |z-2| &= |3z+1| + |3-2z| + |2-z| \\ &\geq |3z+1+3-2z+2-z| = 6.\end{aligned}$$

For equality to hold in the triangle inequality, there should exist a complex number w and positive real numbers r_1, r_2, and r_3 such that

$$\begin{aligned}3z+1 &= r_1 w \\ 3-2z &= r_2 w \\ 2-z &= r_3 w.\end{aligned}$$

Substituting w from the first equation into the second we deduce that z is a real number. In this case one has equality in the triangle inequality if and only if $z \geq -\frac{1}{3}$, $z \leq \frac{3}{2}$ and $z \leq 2$. This means that the solution to the given equation consists of the interval $[-\frac{1}{3}, \frac{3}{2}]$ on the real axis.

2. Find a polynomial with integer coefficients that has the root

$$\sqrt{2} + \sqrt{3} + \sqrt{5}.$$

Solution I: Set $\sqrt{2}+\sqrt{3}+\sqrt{5} = x$. Then $x - \sqrt{5} = \sqrt{2}+\sqrt{3}$, so $x^2 - 2x\sqrt{5} = 2\sqrt{6}$. Squaring again we obtain $x^4 - 4x^3\sqrt{5} + 20x^2 = 24$. It follows that $(x^4 + 20x^2 - 24)^2 = 4x^3\sqrt{5}$, so $x^8 - 40x^6 + 352x^4 - 960x^2 + 576 = 0$, which is the desired polynomial.

Solution II: A polynomial with the given property is

$$\prod(x \pm \sqrt{2} \pm \sqrt{3} \pm \sqrt{5})$$

where the product is taken over the 8 possible choices of the signs plus and minus.

3. There are n^2 people, each on one square of an n-by-n chessboard. Some of them are "infected". At each step, anyone who is infected remains infected, and any healthy person with at least two infected neighbors (corners do not count as neighbors) becomes infected in the next step. Notice that if the main diagonal starts out infected, then after $n - 1$ steps everyone is infected.

   Prove that at least n squares must be infected initially to eventually infect everyone.

   Solution: Consider each side of an infected square that doesn't face another infected square. Call these contagious sides. At the end, when all squares are infected, there are $4n$ contagious sides, just the sides on the edges of the board. Then look at the ways squares can combine to infect other squares. In each situation, it is clear that the number of contagious sides can not increase and will often decrease. Thus, the number of contagious sides stays the same or decreases with each generation. Thus a successful initial situation must start with at least $4n$ contagious sides, so there must be n initially infected squares, and the n squares down the diagonal will work.

4. Note the equality $2^4 = 4^2$. Do there exist other positive integers m and n, $m \neq n$, such that

$$m^n = n^m?$$

Solution: Rewrite the equality as

$$n^{\frac{1}{n}} = m^{\frac{1}{m}},$$

and consider the function $f : (0, \infty) \to (0, \infty)$,

$$f(x) = x^{\frac{1}{x}} = e^{\frac{\ln x}{x}}.$$

Then

$$f' \frac{\ln x}{x} \cdot \frac{1 - \ln x}{x^2},$$

which is positive if $x < e$ and negative if $x > e$. This implies that f is strictly increasing on $(0, e)$ and strictly decreasing on (e, ∞). Also

$\lim_{x\to\infty} f(x) = e^0 = 1$, which is equal to $f(1)$. It follows that the equality from the statement can only hold once, namely for $m = 2 < e$ and its counterpart $n = 4 > e$.

5. Find all possible values of the positive real number a such that the series

$$\sum_{n=1}^{\infty} \cos\left(\pi\sqrt{n^2+an+1}\right)$$

converges.

Solution: We have

$$\cos\left(\pi\sqrt{n^2+an+1}\right) = (-1)^n \cos\left(\pi\sqrt{n^2+an+1} - \pi n\right)$$
$$= (-1)^n \cos\left(\pi \frac{an+1}{\sqrt{n^2+an+1}+n}\right)$$

Note that

$$\lim_{n\to\infty} \frac{an+1}{\sqrt{n^2+an+1}+n} = \frac{a}{2},$$

It follows that in order for the general term of the series to converge to 0, a must be an odd integer.

We observe that $\frac{an+1}{\sqrt{n^2+an+1}+n}$ decreases to $\frac{a}{2}$ if $a = 1$ and increases to $\frac{a}{2}$ for a≥ 2. This is easily seen by differentiating $f(x) = \sqrt{x^2+ax+1} - x$ for $x \geq 0$. If $a \geq 2$, $f'(x) = \frac{x+\frac{a}{2}}{\sqrt{x^2+ax+1}} - 1 \geq \frac{x+\frac{a}{2}}{\sqrt{(x+\frac{a}{2})^2}} - 1 = 0$. And, if $a = 1$, $f'(x) = \frac{x+\frac{1}{2}}{\sqrt{x^2+x+1}} - 1 < \frac{x+\frac{1}{2}}{\sqrt{(x+\frac{1}{2})^2}} - 1 = 0$. This implies that from a moment on, the sequence $\cos\left(\pi\frac{an+1}{\sqrt{n^2+an+1}+n}\right)$ becomes either a decreasing sequence of positive numbers converging to zero, or an increasing sequence of negative numbers converging to zero. The convergence of the series follows then from the alternating series test.

We conclude that the series converges if and only if a is an odd positive integer.

6. Slice a regular tetrahedron into 1000 polyhedra of equal volume by the planes $\pi_1, \pi_2, \ldots, \pi_{999}$ parallel to the base. Also, slice the three faces of the tetrahedron other than the base into polygons of equal area by the planes $\pi'_1, \pi'_2, \ldots, \pi'_{999}$ parallel to the base. What is the number of planes that belong to both of the sets

$$\{\pi_1, \pi_2, \ldots, \pi_{999}\} \text{ and } \{\pi'_1, \pi'_2, \ldots, \pi'_{999}\}?$$

Solution: Each such plane, taken separately, cuts a tetrahedron off the original tetrahedron. Let r_i, respectively r'_i be the similarity ratio between the tetrahedron cut off by the planes π_i respectively π'_i. If the planes π_i and π'_i are listed in order, then the conditions from the statement translate to

$$r_i^3 = \frac{1000 - i}{1000}, \quad r_j'^2 = \frac{1000 - j}{1000},$$

for $i, j = 1, \ldots, 999$. The planes π_i and π'_j coincide if $r_i = r'_j$, and that translates to

$$\sqrt[3]{\frac{1000 - i}{1000}} = \sqrt{\frac{1000 - j}{1000}}.$$

This is equivalent to $1000(1000 - i)^2 = (1000 - j)^3$. We see that j has to be a multiple of 10, say $j = 10k$. The equality becomes $(1000 - i)^2 = (100 - k)^3$. For this to happen, both sides must be sixth powers. In particular $100 - k$ should be a perfect square. There are only 9 possibilities, namely

$$k = 99, 96, 91, 84, 75, 64, 51, 36, 19$$

Each yields one possible value for i. We conclude that the answer to the problem is 9.

7. Show that there are infinitely many positive integers n such that the greatest common divisor of the binomial coefficients

$$\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n-1}$$

is 1.

Solution: Choose $n = p_1 p_2$ where p_1 and p_2 are distinct prime numbers. Then

$$\binom{n}{p_1} = \frac{n(n-1)\cdots(n-p_1+1)}{p_1!},$$

which is not divisible by p_1 because the only factor of p_1 in the numerator cancels with the p_1 in the denominator. For the same reason $\binom{n}{p_2}$ is not divisible by p_2. But $\binom{n}{1} = p_1 p_2$, so the greatest common divisor of the three binomial coefficients $\binom{n}{1}$, $\binom{n}{p_1}$ and $\binom{n}{p_2}$ is 1. Hence n has the desired property. Taking infinitely many pairs of primes we obtain infinitely many n with the required property.

8. Does there exist a positive integer n such that the quadratic equation

$$(n^3 - n + 1)x^2 - (n^5 - n + 1)x - (n^7 - n + 1) = 0$$

has rational solutions?

Solution: Assume for some n, $x = p/q$ is a rational solution, with $\gcd(p, q) = 1$. Then

$$(n^3 - n + 1)p^2 - (n^5 - n + 1)pq - (n^7 - n + 1)q^2 = 0.$$

Each of the coefficients $(n^3 - n + 1), (n^5 - n + 1), (n^7 - n + 1)$ is odd, and of the numbers p and q, either both are odd, or one is odd and the other is even. It follows that the expression on the left is an odd number, which cannot be equal to 0. Hence the answer to the problem is negative.

9. Let A, B, C be $n \times n$ matrices with the property that

$$A + B + C = -I_n,$$

where I_n denotes the $n \times n$ identity matrix. Show that if the matrix

$$AB + AC + BC + ABC$$

is invertible, then the matrix

$$CB + CA + BA + CBA$$

is also invertible.

Solution: Using the hypothesis of the problem we can write

$$\begin{aligned} AB + AC + BC + ABC &= I_n + A + B + C + AB + AC + BC + ABC \\ &= (I_n + A)(I_n + B)(I_n + C). \end{aligned}$$

Because this matrix is invertible, each of the matrices $I_n + A$, $I_n + B$, and $I_n + C$ is invertible as well. But then the product $(I_n + C)(I_n + B)(I_n + A)$ is also invertible. This product equals

$$I_n + A + B + C + CB + CA + BA + CBA = CB + CA + BA + CBA,$$

and we are done.

10. Show that from a group of 2010 people one can select one hundred 5-member teams in such a way that the five members of each team have the same birthday. (Each person can be a member of at most one team.)

Solution I: The solution works the same regardless of whether we take into account leap years or not. The solution we describe below takes into account leap years.

Because $2010 > 4 \cdot 366 = 1464$, by the pigeonhole principle there will be five people in our group with the same birthday. Put them on the first team. There are 2005 people left, and because this number is again greater than 1464 we can select a second team of five with the same birthday. If we repeat the selection k times, there will be $2010 - 5k$ people left, and as long as $2010 - 5k > 1464$ we can continue the process. This means that we can select at least 110 teams and the problem is solved.

Solution II: (By finite induction on the number k of 5-member, same-birthday teams.) Since $2010 > 4 \cdot 366 = 1464$, it is clear that $k = 1$ team, all with the same birthday, can be formed (pigeonhole principle). Assume that $k = 99$ such teams can be formed. The number of members remaining is then $2010 - 5 \cdot 99 = 1515$, which remains greater than 1464, so the 100th team can be formed as well.

Nineteenth Annual Konhauser Problemfest

St. Olaf College **February 26, 2011**

Problems by Andy Niedermaier (Jane Street Capital, New York City)

Calculators of any sort are allowed, although complete justifications are expected, not just the statement of an answer. Use of cell phones or computers is not permitted. Partial credit will be given for progress toward a solution or the answer to part of a question.

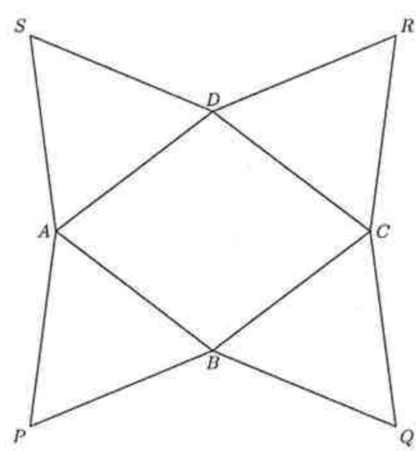
1. **Smallinomial** The cubic equation $ax^3 + bx^2 + cx + d = 0$ has non-zero integer coefficients and distinct integer solutions. Find the smallest possible value for $|a| + |b| + |c| + |d|$.

2. **The Traveling Ant** An ant walking on the plane departs from $(0, 0)$, traveling between lattice points. From any given lattice point (x, y), the ant randomly decides to travel to $(x + 1, y), (x, y + 1)$, or $(x + 1, y + 1)$. After some time, the ant arrives at $(4, 4)$. What is the probability that the ant stopped by $(2, 2)$ along the way?

3. **Fickle Factorial** Find all integers n, $2 \leq n \leq 4010$, such that there exist integers x and y satisfying $2011! = \dfrac{x^2y^3}{n!}$.

4. **Determine It** Let A be a 2011×2011 matrix whose i, j entry is $(-1)^{i+j}$, and let I be the 2011×2011 identity matrix. Let $B(x) = A + Ix$, for $x \in \mathbb{R}$. Compute the determinant of $B(x)$.

5. **Rhomboctagon** $ABCD$ is a rhombus with side length 13. Equilateral triangles are erected on all four sides, resulting in the concave octagon $APBQCRDS$. Given $PQ = 24$, compute the area of the octagon.



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SOLUTIONS

1. **Smallinomial** The cubic equation $ax^3 + bx^2 + cx + d = 0$ has non-zero integer coefficients and distinct integer solutions. Find the smallest possible value for $|a| + |b| + |c| + |d|$.

   Solution. The smallest possible value is 6.

   Let $f(x) = ax^3 + bx^2 + cx + d$, and set $S = |a| + |b| + |c| + |d|$. $S = 6$ is satisfied with $f(x) = (x+1)(x-1)(x-2) = x^3 - 2x^2 - x + 2$. We need to show that the sum cannot be made smaller with some different polynomial $f(x)$.

   Since $f(x)$ is required to have non-zero coefficients, $S \geq 4$. Furthermore, since $f(x)$ is required to have distinct integer solutions, $|d| \geq 2$, hence $S \geq 5$. All that remains is to show that no polynomial of the form $x^3 \pm x^2 \pm x \pm 2$ has distinct integer-valued zeroes.

   To show this, first note that the zeroes would need to be $1, -1$, and one of $\{-2, 2\}$. (In order to get $|d| = 2$.) But $x^3 \pm x^2 \pm x \pm 2$ is odd for $x = \pm 1$, so these cannot be zeroes. □

4. Determine It Let A be a 2011×2011 matrix whose i, j entry is $(-1)^{i+j}$, and let I_{2011} be the 2011×2011 identity matrix. Let $B(x) = A + Ix$, for $x \in \mathbb{R}$. Compute the determinant of $B(x)$.

Solution. Originally we have

$$|B(x)| = \begin{vmatrix} x+1 & -1 & +1 & -1 & \cdots & +1 \\ -1 & x+1 & -1 & +1 & \cdots & -1 \\ +1 & -1 & x+1 & -1 & \cdots & +1 \\ -1 & +1 & -1 & x+1 & \cdots & +1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ +1 & -1 & +1 & -1 & \cdots & x+1 \end{vmatrix}$$

Adding rows does not change the determinant, so add row $k+1$ to row k, for $k = 1, 2, \ldots 2010$:

$$|B(x)| = \begin{vmatrix} x & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & x & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & x & \cdots & 0 & 0 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & x \\ +1 & -1 & +1 & -1 & \cdots & -1 & x+1 \end{vmatrix}$$

Subtracting columns does not change the determinant, so subtract row k from row $k+1$, for $k = 1, 2 \ldots, 2010$:

$$|B(x)| = \begin{vmatrix} x & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 0 \\ +1 & -2 & +3 & -4 & \cdots & -2010 & x+2011 \end{vmatrix}$$

$$= \boxed{x^{2010}(x + 2011)}$$

□

Source: International Math Competition for University Students (9th Edition)

$[PQRS] = 24(12\sqrt{3}+5) = 288\sqrt{5}+120$. We also have $[BPY] = 30$, so now we need to compute $[APZ]$.

$$[APZ] = \frac{13^2}{2}\cos\beta\sin\beta = \frac{13^2}{2}\cdot\frac{12\sqrt{3}+5}{26}\cdot\frac{12-5\sqrt{3}}{26} = \frac{119\sqrt{3}-120}{8}.$$

Putting it all together:

$$\begin{aligned}[APBQCRDS] &= [PQRS] - 4\cdot[BPY] - 4\cdot[APZ] \\ &= (288\sqrt{3}+120) - 4\cdot 30 - 4\cdot\left(\frac{119\sqrt{3}-120}{8}\right) \\ &= 60 + \frac{457\sqrt{3}}{2}\end{aligned}$$

6. Lntegral Compute the exact value of $\int_0^{\pi/2} \ln\cos x \, dx$.

Solution. Let $I = \int_0^{\pi/2} \ln\cos x \, dx$, and note that if we substitute $u = \frac{\pi}{2} - x$ we get $I = \int_0^{\pi/2} \ln\sin x \, dx$. By symmetry, $2I = \int_0^{\pi} \ln\sin x \, dx$, from which we get

$$\begin{aligned}2I &= \int_0^{\pi} \ln\sin x \, dx \\ &= \int_0^{\pi} \ln\sin\left(2\cdot\frac{x}{2}\right) dx \\ &= \int_0^{\pi} \ln\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right) dx \\ &= \int_0^{\pi} \ln 2 + \ln\sin\frac{x}{2} + \ln\cos\frac{x}{2} \, dx \\ &= \pi\ln 2 + 2\left(\int_0^{\pi/2} \ln\sin u + \ln\cos u \, du\right) \\ &= \pi\ln 2 + 4I,\end{aligned}$$

from which we get $\boxed{I = -\frac{\pi\ln 2}{2}}$ □

Since $f(0) = 0$, we get $C = -\frac{\ln(100)}{10000}$. Solving $f(t) = 1$, we get $t = \frac{e^{10000}-1}{100}$. So **yes**, the caterpillar will reach the truck, but not until long after the earth crashes into the sun. □

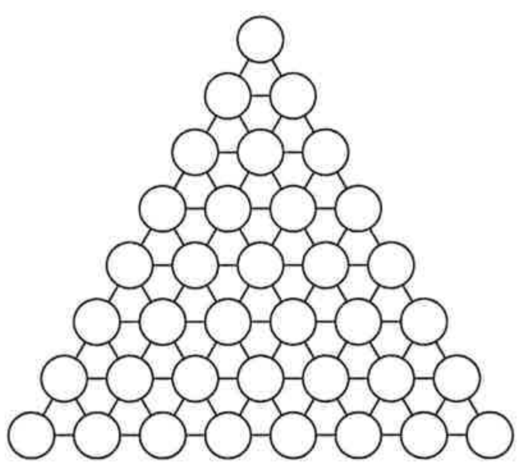
9. Rectangles Galore Let S be an infinite set of rectangles that have one corner at $(0,0)$ and their opposite corner at a point with positive integer coordinates. Show that there must exist rectangles $A, B \in S$ for which the interior of A is contained in the interior of B.

Solution. Suppose no pair of boxes exists. For a box $A \in S$, let $P(A)$ be the corner of A opposite the origin. Suppose box A is the box for which $P(A)$ has a minimal x-coordinate. Then the y-coordinate of $P(A)$ must be maximal over S, and from here we again build a chain of boxes $A, A_1, A_2, \ldots$ such that the x-coordinates of $P(A_n)$ are strictly increasing, and the y-coordinates are strictly decreasing, which contradicts the condition that $|S| = \infty$.

****Note:** This problem was intended to use rectangles whose sides are parallel to the x and y-axes. This was not stated in the given problem, so there does exist an infinite set of rectangles for which the interior of no rectangle is contained in the interior of any other rectangle. For example, consider the set of rectangles whose corners are at $(0,0), (n,1), (n-1,n+1), (-1,n)$ for all integers $n > 0$.

Source: Brazilian Math Olympiad

10. Number Puzzle The triangular lattice below contains 8 rows and 36 cells. Consider all ways in which the integers $1, 2, \ldots, 6$ can be filled into the lattice so that each integer appears exactly 6 times. Find the largest n for which the following statement is true: There must exist a straight line (parallel to a side of the triangle) that contains at least n different integers.



Solution. The largest n is $n = 3$. We will first show n is at least 3 for any arrangement, then we will provide an example to show that $n = 3$ is achievable.